INSTANTONS IN FINITE VOLUME, QUANTUM TUNNELLING AND COSMIC BOUNCE

Silvia Pla (silvia.pla_garcia@kcl.ac.uk)
In collaboration with Wenyuan Ai, Jean Alexandre, Drew Backhouse, Matthias Carosi, Katy Clough, and Björn Garbrecht

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Theoretical Particle Physics and Cosmology, King's College London, WC2R 2LS, UK

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Abstract: Tunnelling between two degenerate vacua is allowed in finite-volume Quantum Field Theory. This effect induces a non-trivial vacuum energy, which result from the competition of different saddle point configurations in the partition function. In this talk, I will describe this mechanism and discuss its relevance to induce a cosmological bounce.

Part I

Introduction, assumptions and main results

QFT in curved spaces studies the behaviour of quantum matter fields as *test fields* propagating in a specific background.

A second step to better understand the interaction between quantum fields and gravity is to study the *backreaction problem*, i.e., the effect of quantum fields on the background metric.

Semiclassical gravity.

We have to solve the semiclassical Einstein equations in a self-consistent way:

$$\kappa^{-1}\text{``}G_{\mu\nu}\text{''} = T_{\mu\nu}^{\text{class}} + \langle T_{\mu\nu}\rangle_{\text{ren}}$$

Key concept: The Null Energy Condition

Singularity theorems. They show that singularities (in cosmology) are inevitable under very general circumstances.

However, they have in their assumptions a restriction on the energy-momentum tensor.

The **Null Energy Condition** (NEC)

$$T_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0\,,$$

where ℓ^{μ} is a null vector. For a perfect fluid the null energy condition reads

$$\rho + p \geq 0\,.$$

- *Energy conditions can be violated by quantum fields.*

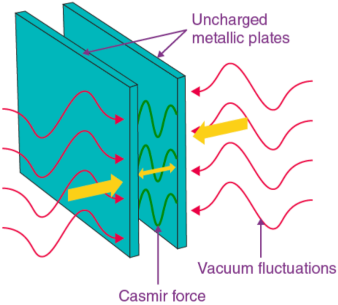
If the NEC is violated $\rightarrow$ can we avoid the “initial singularity”?

Example: The Casimir effect

$$E_{\text{cas}} \equiv E_{\text{discrete}} - E_{\text{continuum}}$$

Electromagnetic ground energy, massless scalar field.

$$E_{\text{cas}} = -\frac{\pi^2 \mathcal{A}}{720 a^3} \,,$$



$$\begin{aligned} \rho &= \frac{E_{\text{cas}}}{a\,\mathcal{A}} = -\frac{\pi^2}{720 a^4} \,, \\ p &= -\frac{1}{\mathcal{A}} \frac{\partial E_{\text{cas}}}{\partial a} = -\frac{\pi^2}{240 a^4} \end{aligned}$$

NEC violation:

$$\rho + p < 0 \,.$$

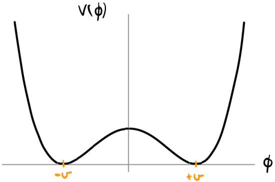
In finite volume, there is another affect:

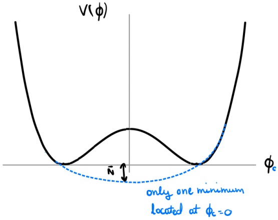
NEC violation via tunnelling

Real scalar field with a double-well bare potential with two degenerate minima in *finite volume* V_0.

$$U(\phi) = \frac{\lambda}{4!}(\phi^2 - v^2)^2 + \kappa^{-1}\Lambda\,.$$

In this context, **quantum tunneling** can happen between the two minima. **Symmetry restoration**; convexity. Only one ground/vacuum state → We are interested in the vacuum energy.





The effective potential reads:

$$U_{eff}(\phi_c) = U_0 + M^2\phi_c^2 + \mathcal{O}(\phi_c^4)\,, \qquad \alpha^3 = \frac{S_0}{\hbar} \quad \text{and} \quad S_0 \sim L_0^3$$

$$U_0 = \kappa^{-1}\Lambda\left(1 - r\,\frac{e^{-\alpha^3}}{\alpha^{3/2}}\right) \qquad r = \frac{\lambda\kappa v^4}{3\sqrt{3\pi}\Lambda}$$

From this result, we can compute the stress-energy tensor of the ground state: the Null Energy Condition (NEC) is violated

$$\frac{\kappa}{\Lambda}(\rho + p) = -r\,e^{-\alpha^3}\left(\alpha^{3/2} + \frac{1}{2}\alpha^{-3/2}\right) < 0\,.$$

We can couple this effect to gravity via the Einstein equations.

[FLRW metric; homogeneous ground state fluid $\rightarrow$ perfect fluid]

If we start from a contracting phase $H < 0$, we can induce a bounce.

Assumptions, methods, and limitations

- **Finite Volume** $\rightarrow$ allows tunnelling [*3-torus, 3-sphere, box...*].
  - For simplicity we consider periodic boundary conditions. Fundamental volume cell $V_0 = L_0^3$. Comoving volume $a^3 L_0^3$.
  - We neglect spatial curvature K.

- **Adiabatic approximation.** Expansion rate $\frac{\dot{a}}{a} \ll$ tunneling rate (intantaneous effect).

  We set $a = cte$ for the computation of the effective theory.
  We couple the effective theory to gravity restoring $a = a(t)$.

- **Equilibrium QFT techniques:** Path integral approach in the $\langle in | out \rangle$ formalism. **Wick rotation**. Euclidean/imaginary time is needed to study this effect.

  Naively $t \rightarrow \tau = it\,;\, \sqrt{-g} \rightarrow \sqrt{g_E}\,;\, iS \rightarrow -S_E ...$

Part II

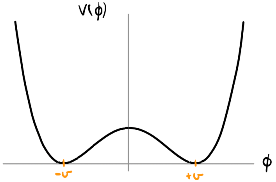
The effective theory

We start from the classical theory:

required for renormalization.

$$V(\phi) = \frac{\lambda}{4!}(\phi^2 - v^2)^2 + \bar{\Lambda} + j\phi$$

essential to obtain the effective action.



How do we arrive to U_0?

- We should find the partition function $Z[j]$ describing the system.
  - Saddle point expansion (one-loop expansion)

$$\phi = \phi_s + \delta\phi\,; \quad \phi_s \equiv \text{classical solution} \quad \delta\phi \equiv \text{quantum fluctuation}$$

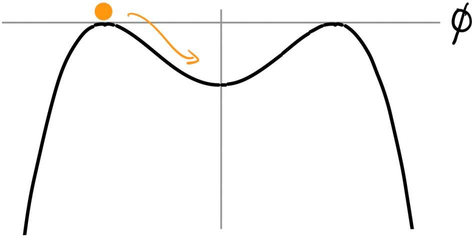
  - Semiclassical approximation: *several saddle points* ϕ_n are relevant for our analysis. We sum over all relevant configurations.

$$Z_E[j] = \int \mathcal{D}[\phi]\, e^{-S[\phi]/\hbar} \simeq \sum_n Z_n\,; \qquad Z_n = F_n\, e^{-S[\phi_n]/\hbar}\,.$$

Relevant saddle points.

- Static saddle points.
- Instantons [gas of instantons; dilute gas approximation].

In imaginary time $U(\phi) \to -U(\phi)$. "Upside down potential".



Note: we focus on homogeneous configurations $\phi_s = \phi_s(t)$.

$$\phi'' + \omega^2 \phi - \frac{\omega^2}{v^2}\phi^3 = j\,, \qquad \omega^2 = \frac{\lambda v^2}{12} = \frac{m^2}{2}$$

Let's focus on $j = 0$ (the vacuum).

- **Static Saddle points.**

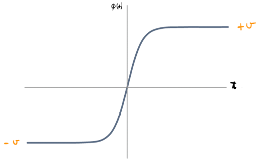
$$\phi_{1,2} = \pm v \qquad S_{1,2} = a^3 L_0^3 T \bar{\Lambda} \,.$$

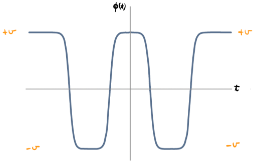
- **Instantons.** The basic solution is

$$\phi_i = \pm v \tanh(\omega(t - t_1)) \,, \qquad S = a^3 S_0 \propto a^3 L_0^3$$

  We can also have multiple jumps

$$\phi^{(p)} \simeq v^p \tanh(\omega(t - t_1)) \tanh(\omega(t - t_2)) \cdots \tanh(\omega(t - t_p)) \,, \qquad S = p a^3 S_0$$





Relevant saddle points

- We have to sum over all the classical configurations.

$$Z_E = [v] + [-v] + [K] + [\bar{K}] + [K\bar{K}] + [\bar{K}K] + [K\bar{K}K] + \cdots$$

where

$$[v] = [-v] = e^{-a^3 L_0^3 T\bar{\Lambda} + TE_{\text{cas}}(L)}\,, \qquad [K] = \int_{T/2}^{T/2} dt_1 [I][v] = T[I][v]$$

and with

$$[I] = \sqrt{\frac{a^3 S_0}{2\pi}}\, e^{-a^3 S_0} \left(\frac{\det' S_E''[\phi_i]}{\det S_E''[v]} \right)^{-\frac{1}{2}} \simeq \sqrt{12}\,\omega \sqrt{\frac{a^3 S_0}{2\pi}}\, e^{-a^3 S_0}\,.$$

- For N-jumps we find

$$\left[K_1 \overline{K}_2 \ldots K_{N-1} \overline{K}_N\right] \approx \frac{T^N}{N!} [I]^N [v]\,,$$

- **The partition function can be resummed!**

$$Z_E = 2[v] e^{-T[I]}$$

The vacuum energy is obtained for $j = 0$ [assuming $a = 1$]:

$$E_0 = -\lim_{T\to\infty} \frac{1}{T} \ln Z_E[0] = E_{\rm stat} - [I]$$

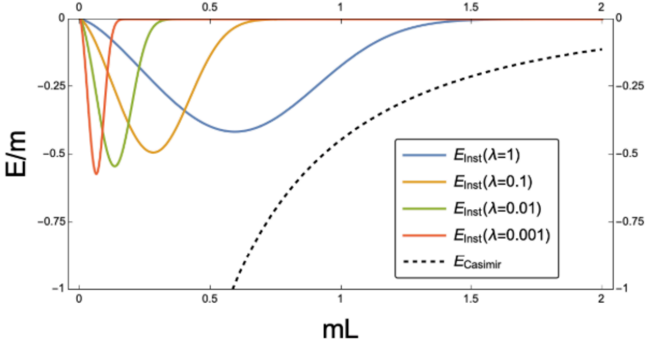
where

$$\begin{aligned} E_{\rm stat} &= -\lim_{T\to\infty} \frac{1}{T} \log[v] = V_0 \bar{\Lambda} + E_{\rm casimir}\,, \\ [I] &\simeq \sqrt{12}\omega \sqrt{\frac{S_0}{2\pi}}\, e^{-S_0}\,, \qquad S_0 \propto V_0 = L_0^3\,. \end{aligned}$$

with

$$E_{\rm casimir} = \frac{1}{2}\sum_{\mathbf{n}} \sqrt{\mathbf{k}_n^2 + m^2} - \frac{V_0}{2}\int_{\mathbb{R}^3} \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{\mathbf{k}^2 + m^2}$$

This result is strongly dependent on the geometry/topology. For a 3-torus (periodic boundary conditions) $E_{\rm casimir} < 0$ and bigger than $[I]$.



For the three torus E_{casimir} is always negative. Asymptotically

$$E_{\text{casimir}} \rightarrow -\frac{\beta}{L} \qquad \text{for} \qquad mL \ll 1$$

$$E_{\text{casimir}} \rightarrow -\frac{\gamma}{L}\sqrt{(mL)^3}\, e^{-mL} \qquad \text{for} \qquad mL \gg 1$$

The thermodynamic pressure and energy density (at zero temperature) can be defined as

$$\rho \equiv \frac{1}{V} F_{\text{true}} \,,$$
$$p \equiv -\frac{\partial F_{\text{true}}}{\partial V} \,.$$

where $F_{\text{true}} = E_0 = E_{\text{stat}} - [I]$.

The isothermal compressibility

$$K \equiv -\frac{1}{V}\frac{\partial V}{\partial p} \equiv \frac{1}{V}\left(\frac{\partial^2 F_{\text{true}}}{\partial V^2}\right)^{-1}$$

is negative for all V.

This is a sign of instability!

Part III

Towards a cosmic bounce

Concrete example: Let us forget about the Casimir energy.

We have derived the properties of the quantum vacuum from the minimum of the effective action, i.e., the minimum of the potential

$$U_{eff}(\phi_c = 0) = \kappa^{-1}\Lambda\left(1 - r\frac{e^{-\alpha^3}}{\alpha^{3/2}}\right) ,$$

with

$$\alpha^3 = a^3\frac{S_0}{\hbar} \quad \text{and} \qquad r = \frac{\lambda\kappa v^4}{3\sqrt{3\pi}\Lambda} .$$

In particular, we can obtain the *energy density and the pressure*

$$\frac{\kappa\rho}{\Lambda} = \tilde{\rho} = +1 - r\frac{e^{-\alpha^3}}{\alpha^{3/2}}$$

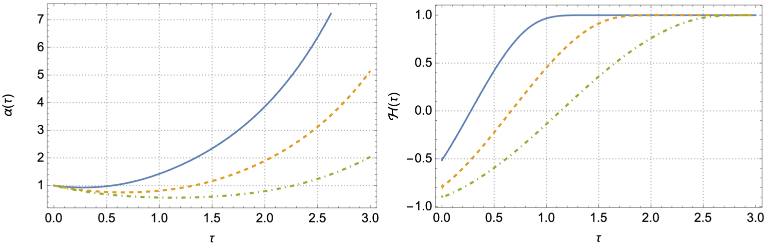
$$\frac{\kappa p}{\Lambda} = \tilde{p} = -1 - re^{-\alpha^3}\left(\alpha^{3/2} - \frac{\alpha^{-3/2}}{2}\right) .$$

We can couple this ground state fluid to the Einstein equations and study its consequences.

Friedmann equations

$$H^2 = \frac{\kappa}{3}\rho \quad \text{(constraint)}; \qquad \frac{\ddot{a}}{a} = -\frac{\kappa}{6}(\rho + 3p) \quad \text{(dynamics)};$$

We solved the Friedmann equations for different values of r, and focusing on $H(t_0) < 0$ (initial contracting phase).



Note: If a bounce occurs, then $H = 0$ and $H' > 0$. This implies $\rho = 0$ and $\rho + 3p < 0$.

Question: How robust is this solution?

We can add a *second component* to test it. We focus on the case of anisotropy, since it is the component that dominates the energy budget the quickest during a collapse. Bianchi-I metric:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a_1^2\,\mathrm{d}x^2 + a_2^2\,\mathrm{d}y^2 + a_3^2\,\mathrm{d}z^2$$

We work with averaged quantities: $a^3 = a_1 a_2 a_3$

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right)\,, \quad \sigma^2 = \frac{1}{18}\left[(H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_1 - H_3)^2\right]$$

Note: ρ and p remain the same, with the change $\alpha_{iso} \to \alpha = \sqrt{\alpha_1\alpha_2\alpha_3}$.

Friedmann equations

$$H^2 = \frac{\kappa}{3}\rho + \sigma^2\,; \qquad \dot{H} + H^2 = -\frac{\kappa}{6}(\rho + 3p) - 2\sigma^2$$

It can be shown that $\sigma^2 \sim a^{-6}$. The anisotropy plays a role of a homogeneous fluid with equation of state $w = 1$ (i.e., $\rho_a = p_a = 3\kappa^{-1}\sigma^2$).

A critical solution $(\rho_c, p_c, a_c, \sigma_c^2)$ can be found by imposing the condition

$$\boxed{H = H' = 0}$$

$$\tilde{p}_c = \tilde{\rho}_c = -\tilde{\sigma}_c^2$$
$$\downarrow$$
$$4\alpha_c^{3/2} + re^{-\alpha_c^3}\left(-3 + 2\alpha_c^3\right) = 0$$

This solution gives *the minimum value α can have* for the bounce to succeed.

The critical point is unstable: a value of α that is slightly larger than α_c leads to a bounce and a value which is slightly smaller leads to a collapse.

We can understand this instability in terms of the *total sum* $\rho_{tot} + p_{tot}$.

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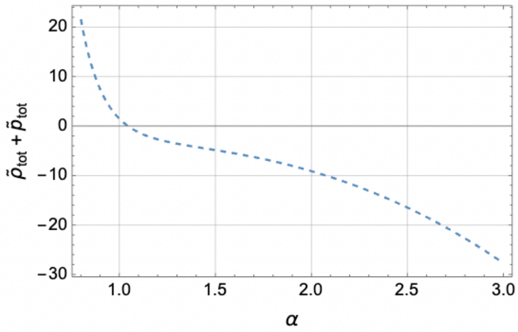
First, from $H = 0$ we get,

$$\rho_{tot} = 0\,, \quad \rightarrow \quad \rho + \rho_a = 0 \quad \rightarrow \quad 1 - r\,\frac{e^{-\alpha^3}}{\alpha^{3/2}} + \frac{s}{\alpha^6} = 0\,.$$

Imposing this constraint, the sum $\rho + p$ reads:



- We studied the effect of finite volume on the vacuum energy E_0 of a scalar field in a double-well potential. In this context, two effects emerge: the Casimir effect (E_{casimir}) and quantum tunnelling ([I]).
- For a 3-torus, both contributions violate the null energy condition. Furthermore, the Casimir effect dominates over [I]. This result strongly depends on the geometry and boundary conditions.
- This effect can induce a bounce when coupled to gravity. However, the bounce is not always guaranteed in more involved configurations.

Possible extensions:

- Systematic analysis of other geometries and boundary conditions.
- Gravitational collapse.
- Extra dimensions $\rightarrow$ NEC violation and stability of the compact dimensions.

THANKS FOR YOUR ATTENTION

The approximated result I showed during the talk is:

$$[I] = \sqrt{12m^2}\sqrt{\frac{m^3 L^3}{\pi\lambda}} \exp\left\{-\frac{2m^3 L^3}{\lambda}\right\} .$$

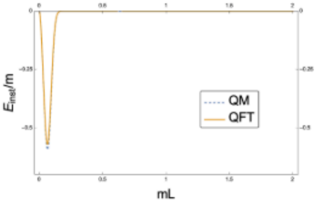
This result has been obtained ignoring the spatial gradients in the fluctuation determinant. Introducing these corrections we obtain the following result:

$$[I] = -\sqrt{12m^2}\sqrt{\frac{m^3 L^3}{\pi\lambda}} \exp\left\{-\frac{2m^3 L^3}{\lambda}\right\} \exp\left\{G_{(3)}\right\}$$

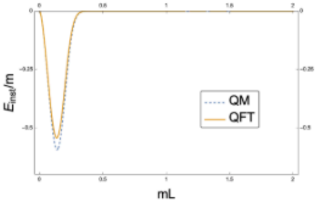
where

$$G_{(3)} = + \sum_{\mathbf{n}\in\mathbb{Z}^3}{}' e^{-\frac{k_n^2}{\Lambda^2}} \log \frac{3m^2 + 2k_n^2 + 3m\sqrt{k_n^2 + m^2}}{k_n\sqrt{4k_n^2 + 3m^2}} - \frac{3m^3 L^3}{32\pi^2}\left(-21 + \gamma_E + \frac{4\Lambda^2}{m^2} - \log\frac{4\Lambda^2}{m^2}\right)\Bigg\}$$

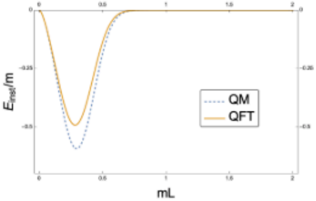
Here I compare the approx. and the exact result for different values of λ.



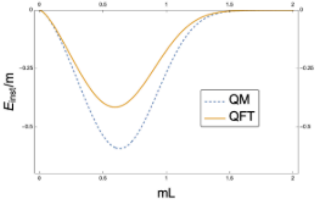
(a) $\lambda = 0.001$



(b) $\lambda = 0.01$



(c) $\lambda = 0.1$



(d) $\lambda = 1$

From $Z_E[0]$, we can find the effective action at the ground state (i.e., the zero-point energy)

$$\Gamma[0] = -\ln Z_E[0] \simeq \int \mathrm{d}^4x \sqrt{g}\,\bar{\Lambda}\left(1 - r\frac{e^{-\alpha^3}}{\alpha^{3/2}}\right)$$

If we want to obtain the full effective action, need to include j.

- From $Z_E[j]$ we obtain the *effective theory* in terms of an effective field ϕ_c

$$\phi_c[j] = \frac{1}{\sqrt{g}}\frac{\hbar}{Z_E[j]}\frac{\delta Z_E[j]}{\delta j} \qquad \longrightarrow \qquad j[\phi_c]$$

- The effective action $\Gamma[\phi_c]$ is defined through a Legendre transformation as a functional of ϕ_c.

$$\Gamma[\phi_c] = -\hbar\ln(Z_E[j[\phi_c]]) - \int \mathrm{d}^4x\sqrt{g}\,\phi_c\, j[\phi_c]$$

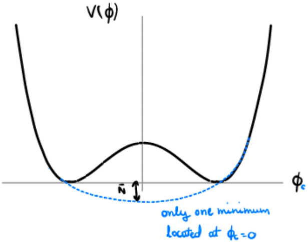
In practice, it is very difficult to obtain $\phi_c[j]$ and invert it. However, for small j we find

$$\phi_c = -M^{-2}\, j + \mathcal{O}(j^3)\,; \qquad \phi_c \text{ is linear in } j!$$

For $j = 0$ we find $\phi_c = 0$. **Symmetry restoration.**

Around the minimum, the effective action reads:

$$\Gamma[\phi_c] = \Gamma[0] + \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g}\,M^2\phi_c^2 + \mathcal{O}(\phi_c^4)\,.$$

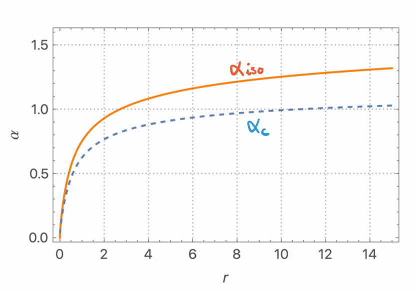


One can also infer a *maximum value* for the rescaled scale factor α_{iso} at the bounce, which happens when the anisotropy is negligible and the quantum field dominates.

$$\tilde{\rho} = 0$$
$$\downarrow$$
$$\alpha_{iso}^{3/2} = r\, e^{-\alpha_{iso}^3}$$

Which leads to the two regimes

$$\alpha_{iso} \sim (\ln r)^{1/3} \quad \text{for} \quad r \gg 1$$
$$\alpha_{iso} \sim r^{2/3} \quad \text{for} \quad r \ll 1\,,$$



The critical solutions allow us to estimate a size of the universe at the bounce. Assuming $\lambda \sim 1$:

$$\frac{\alpha_c(r)}{m} \lesssim L_b \lesssim \frac{\alpha_{iso}(r)}{m}$$

Example: for $r = 1$, if the bounce occurred when $L \sim 1$ m, the scalar field would need a mass of $\sim 10^{-7}$ eV and a vacuum energy Λ of order $10^{-140}\ \ell_p^{-2}$.

Remember:

$$r \sim \frac{\kappa m^4}{\Lambda}$$