Relativistic Entanglement and Perturbative Decoherence

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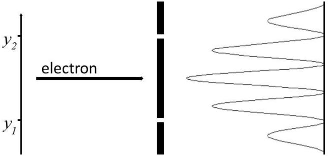
Spatial double slit experiment (Davisson and Germer)

Electron passes slits at $y = y_1$ or $y = y_2$

Spatial superposition at screen $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|y_1\rangle + |y_2\rangle \right)$

Momentum state

$$\langle p|\psi\rangle = \frac{1}{\sqrt{2}} \left(\langle p|y_1\rangle + \langle p|y_2\rangle \right) = \frac{1}{\sqrt{2}} \left(e^{-\frac{i}{\hbar} p y_1} + e^{-\frac{i}{\hbar} p y_2} \right)$$



Temporal double slit experiment (Lindner et al)

Ultra-short laser pulse ionizes atom when $\mathbf{E}(t) = \mathbf{E}_{max}$

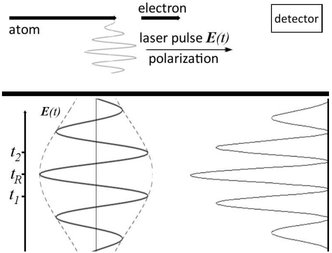
Electron emitted at $t = t_1$ or at $t = t_2$

(Electron emitted at $t = t_R$ moves away from detector)

Temporal superposition $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|t_1\rangle + |t_2\rangle \right)$

Energy state

$$\langle E|\psi\rangle = \frac{1}{\sqrt{2}} \left(\langle E|t_1\rangle + \langle E|t_2\rangle \right) = \frac{1}{\sqrt{2}} \left(e^{-\frac{i}{\hbar} E t_1} + e^{-\frac{i}{\hbar} E t_2} \right)$$



Entanglement and Interference

Temporal interference from entangles electrons (Palacios et al)

Entangled electrons

Sequential double ionization of helium

Electrons emitted at small $\Delta t = t_2 - t_1$ with $\Delta E \; \Delta t > \hbar/2$

Indistinguishable particles in singlet state

Nonrelativistic treatment of singlet state

Antisymmetric under exchange of electrons

$$\psi(t_1, t_2) = \frac{1}{\sqrt{2}} \left[e^{-\frac{i}{\hbar}(E_1 t_1 + E_2 t_2)} + e^{-\frac{i}{\hbar}(E_1 t_2 + E_2 t_1)} \right] \times \text{antisymmetric spin factor } S_{12}$$

$$= \frac{1}{\sqrt{2}} \, e^{-\frac{i}{\hbar} ET} \left[e^{\frac{i}{\hbar} \Delta E \Delta t/2} + e^{-\frac{i}{\hbar} \Delta E \Delta t/2} \right] \times S_{12}$$

$$T = \tfrac{1}{2}\,(t_1 + t_2) \qquad \Delta t = t_2 - t_1 \qquad E = E_1 + E_2 \qquad \Delta E = E_1 - E_2$$

Interference fringes in time domain

Problem: nonrelativistic states defined at different times are **not coherent**

Entanglement and Interference

Requirements for a consistent theory of superposition in time

Space and time on same footing: $(\mathbf{x}, t) \longrightarrow x^\mu$

- States transform under $SL(2,C)$ covering group of $O(3,1)$

Relativistic Hilbert space

- Coherent eigenstates of complete set of operators in a given representation
- Defined with respect to a shared continuous parameterization
  - Basis states with spin $|\sigma, x^\mu, \tau\rangle$ or $|\sigma, p^\mu, \tau\rangle$ defined at given time τ
  - Trajectory-independent evolution parameter τ: $[\tau, \dot{x}^\mu] = 0$

Wigner's induced representation of $SL(2,C)$ over $SU(2)$

- Spin = eigenstate of rotation generators $\in SU(2) \subset SL(2,C)$
- $SU_n(2)$ operates in spacelike hypersurface normal to timelike n^μ (usually $n^\mu = p^\mu$)
- Induced $SL(2,C)$ boosts x^μ, p^μ but rotates spin components in hypersurface

Coherent states must belong to same representation of $SU(2)_n$ $\longrightarrow$ same n^μ

- Many body state — irreducible representation of coherent product states

Stueckelberg-Horwitz-Piron (SHP) Covariant Mechanics

Framework for classical and quantum special and general relativity

External evolution parameter τ advances monotonically

8D phase space $(x^\mu, \dot{x}^\mu) \longrightarrow \dot{x}^2$ unconstrained

Event $x^\mu(\tau)$ can change direction in coordinate time $x^0 \longrightarrow$ antiparticle

Free event described by Lagrangian and equivalent Hamiltonian

$$L_0 = \frac{1}{2} M \dot{x}^\mu \dot{x}_\mu \qquad K_0 = \frac{1}{2M} p^\mu p_\mu \qquad p_\mu = \partial L / \partial \dot{x}^\mu$$

Euler-Lagrange and Hamilton equations

$$0 = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} \qquad \dot{x}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \dot{p}_\mu = -\frac{\partial K}{\partial x_\mu}$$

Stueckelberg-Schrodinger equation with interactions

$$i\partial_\tau \Psi(x,\tau) = K \Psi(x,\tau) \qquad K_0 \longrightarrow K = K_0 + V(x)$$

Horwitz-Piron-Reuse representation of spin

Modified Wigner representation of $SL(2,C)$ from $SU_n(2)$

Induced on arbitrary timelike vector $n^\mu \neq p^\mu$

Stueckelberg-Horwitz-Piron (SHP) Covariant Mechanics

Generalized central force problem

Relativistic bound state solutions

Spinless particles of spacelike separation $x = (t, \mathbf{r})$

$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$

Replace $V(r) \longrightarrow V(\rho) \qquad \rho = \sqrt{\mathbf{r}^2 - t^2}$

To obtain correct spectrum and multiplicity $\longrightarrow$ break spacetime symmetry

- Choose arbitrary spacelike vector s^μ
- Restrict dynamics to subspace of spacelike $\left\{x^\mu \mid x^2 \leq 0 \, , \; x_\perp^2 \leq 0\right\}$
- States ψ_s transform under $O(2,1) \subset O(3,1)$

Induce representation of $O(3,1)$

$O(3,1)$ on $\psi_s \;\longrightarrow\;$ generators containing $(x^\mu, \partial/\partial x^\mu)$ and $(s^\mu, \partial/\partial s^\mu)$

Generators $\rightarrow$ Casimir operators $\rightarrow$ eigenvalues $\rightarrow$ full state characterization

Zeeman and Stark effects (Land)

Dynamical $s^\mu(\tau) \;\longrightarrow\;$ extended phase space $\left\{(x^\mu, \dot{x}^\mu), (s^\mu, \dot{s}^\mu)\right\}$

$O(3,1)$ generators $X^{\mu\nu}$ couple to electromagnetic $F_{\mu\nu}$

Classical extended phase space (extra dimensions)

Gauge theory

Classical phase space and gauge fields

$$\left\{\left(x^\mu,\dot{x}^\mu\right),\left(\zeta^\mu,\dot{\zeta}^\mu\right)\right\} \qquad \left\{A^\mu\left(x,\zeta\right),\chi^\mu\left(x,\zeta\right)\right\}$$

Classical Lagrangian

$$L=\frac{1}{2}M\dot{x}^\mu\dot{x}_\mu+\frac{1}{2}M\dot{\zeta}^\mu\dot{\zeta}_\mu+e\dot{x}_\mu A^\mu\left(x,\zeta\right)+e\dot{\zeta}_\mu\chi^\mu\left(x,\zeta\right)$$

Gauge invariance

$$A^\mu\left(x,\zeta\right)\longrightarrow A^\mu\left(x,\zeta\right)+\frac{\partial\Lambda}{\partial x^\mu} \qquad \chi^\mu\left(x,\zeta\right)\longrightarrow \chi^\mu\left(x,\zeta\right)+\frac{\partial\Lambda}{\partial \zeta^\mu}$$

Variation with respect to x^μ and ζ^μ $\longrightarrow$ Lorentz force

$$M\ddot{x}^\mu(\tau)=eF^{\mu\nu}\left(x,\zeta\right)\dot{x}_\nu(\tau)+eH^{\mu\nu}\left(x,\zeta\right)\dot{\zeta}_\nu(\tau)$$
$$M\ddot{\zeta}^\mu(\tau)=eG^{\mu\nu}\left(x,\zeta\right)\dot{\zeta}_\nu(\tau)-eH^{\mu\nu}\left(x,\zeta\right)\dot{x}_\nu(\tau)$$

Field strengths

$$F^{\mu\nu}=\frac{\partial A^\nu}{\partial x_\mu}-\frac{\partial A^\mu}{\partial x_\nu} \qquad G^{\mu\nu}=\frac{\partial \chi^\nu}{\partial \zeta_\mu}-\frac{\partial \chi^\mu}{\partial \zeta_\nu} \qquad H^{\mu\nu}=\frac{\partial \chi^\nu}{\partial x_\mu}-\frac{\partial A^\mu}{\partial \zeta_\nu}$$

Classical extended phase space (extra dimensions)

Gauge invariant scalar Hamiltonian

Canonical momenta

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = M\dot{x}_\mu + eA_\mu \qquad \longrightarrow \qquad \dot{x}_\mu = \frac{1}{M}\left(p_\mu - eA_\mu\right)$$

$$\pi_\mu = \frac{\partial L}{\partial \dot{\zeta}^\mu} = M\dot{\zeta}_\mu + e\chi_\mu \qquad \longrightarrow \qquad \dot{\zeta}_\mu = \frac{1}{M}\left(\pi_\mu - e\chi_\mu\right)$$

Hamiltonian

$$K = \frac{1}{2M}\left[\left(p^\mu - eA^\mu\right)\left(p_\mu - eA_\mu\right) + \left(\pi^\mu - e\chi^\mu\right)\left(\pi_\mu - e\chi_\mu\right)\right]$$

Horwitz-Piron-Reuse representation of spin

Induced representation on arbitrary timelike unit vector $n^\mu \neq p^\mu$

We identify $n^\mu \longrightarrow n^\mu(\tau) = \pi^\mu(\tau)/M$

Entanglement $\Rightarrow$ particles in same representation of $SU(2)_n$

Requires evolution $n_i^\mu(\tau) = n_j^\mu(\tau)$ for component states i, j

Classical extended phase space (extra dimensions)

Equations of motion

Two classical particles with identical initial conditions

$$\zeta_1^\mu(0) = \zeta_2^\mu(0) \qquad \pi_1^\mu(0) = \pi_2^\mu(0) \longrightarrow n_1^\mu(0) = n_2^\mu(0)$$

Hamilton equations for $\pi_i^\mu(\tau)$

$$\dot{\pi}_1^\mu = \frac{e}{M}\left[\left[p_1^\nu - eA^\nu(x_1,\zeta_1)\right]\frac{\partial A_\nu(x_1,\zeta_1)}{\partial \zeta_\mu^1} + \left[\pi_1^\nu - e\chi^\nu(x_1,\zeta_1)\right]\frac{\partial \chi_\nu(x_1,\zeta_1)}{\partial \zeta_\mu^1}\right]$$

$$\dot{\pi}_2^\mu = \frac{e}{M}\left[\left[p_2^\nu - eA^\nu(x_2,\zeta_2)\right]\frac{\partial A_\nu(x_2,\zeta_2)}{\partial \zeta_\mu^2} + \left[\pi_2^\nu - e\chi^\nu(x_2,\zeta_2)\right]\frac{\partial \chi_\nu(x_2,\zeta_2)}{\partial \zeta_\mu^2}\right]$$

Case 1 $A^\nu(x_1,\zeta) \neq A^\nu(x_2,\zeta)$ or $\chi^\nu(x_1,\zeta) \neq \chi^\nu(x_2,\zeta) \implies \dot{\pi}_1^\mu \neq \dot{\pi}_2^\mu$

Case 2 $A^\mu = A^\mu(x)$ and $\chi^\mu = \chi^\mu(\zeta)$

$$\dot{\pi}_i^\mu = \frac{e}{M}\left[\pi_i^\nu - e\chi^\nu(\zeta_i)\right]\frac{\partial \chi_\nu(\zeta_i)}{\partial \zeta_\mu^i} \implies \dot{\pi}_1^\mu = \dot{\pi}_2^\mu \implies n_1^\mu = n_2^\mu$$

Quantum states: unitary representation of Poincaré group

Lie algebra

Unitary Poincaré transformation $|\psi'\rangle = U(\Lambda, a)|\psi\rangle$ on extended phase space

Lorentz transformation $\Lambda \in O(3,1)$ and translation a

$U(\Lambda, a) \simeq 1 + ia^\mu \left(P_\mu + \Pi_\mu\right) + i\omega^{\mu\nu}\left(L_{\mu\nu} + N_{\mu\nu}\right)$

Two independent sets of Poincaré generators

$$L_{\mu\nu} = \left(X_\mu P_\nu - X_\nu P_\mu\right) \qquad N_{\mu\nu} = \left(\zeta_\mu \Pi_\nu - \zeta_\nu \Pi_\mu\right)$$

$$P_\mu = i\partial/\partial X^\mu \qquad \Pi_\mu = i\partial/\partial \zeta^\mu$$

P^2 unconstrained in SHP framework

$\Pi^2|\psi\rangle = M^2|\psi\rangle$ on states $\longrightarrow$ describe spin on $\left(\zeta_\mu, \Pi_\mu\right)$ sector

Lie algebra

$$\left[L_{\mu\nu}, P_\sigma\right] = i\left(g_{\nu\sigma}P_\mu - g_{\mu\sigma}P_\nu\right)$$

$$[L^{\mu\nu}, L^{\rho\sigma}] = i\left(g^{\nu\rho}L^{\mu\sigma} + g^{\mu\sigma}L^{\nu\rho} - g^{\mu\rho}L^{\nu\sigma} - g^{\nu\sigma}L^{\mu\rho}\right)$$

$$\left[P_\mu, P_\nu\right] = \left[\Pi_\mu, \Pi_\nu\right] = \left[P_\mu, \Pi_\nu\right] = \left[P_\mu, N_{\sigma\nu}\right] = \left[\Pi_\mu, L_{\sigma\nu}\right] = [L^{\mu\nu}, N^{\rho\sigma}] = 0$$

Same relations for Π_μ and $N_{\mu\nu}$

Quantum states: unitary representation of Poincaré group

Standard representation theory for Lorentz group

Partition generators $M_{\mu\nu} = L_{\mu\nu} + N_{\mu\nu}$ into boosts M_{0i} and rotations M_{ij}

Partition inequivalent left and right handed representations of $SU(2)$

Decomposes $SO(3,1) = SU(2)_L \otimes SU(2)_R$

Two components spinors transform under $A^{L,R} \in SL(2,C)$ where

$$A^L = \exp\left(\boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2 + i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}/2\right) \qquad A^R = \exp\left(-\boldsymbol{\beta}\cdot\boldsymbol{\sigma}/2 + i\boldsymbol{\omega}\cdot\boldsymbol{\sigma}/2\right)$$

Inequivalent bases for $SL\left(2,C\right)$ using $\sigma^0 = \mathrm{diag}\left(1,1\right)$ and $\mathsf{C} = i\sigma^2$

$$\sigma^\mu = \left\{\sigma^0, \boldsymbol{\sigma}\right\} \qquad \underline{\sigma}^\mu = \mathsf{C}\sigma^*_\mu \mathsf{C}^\dagger = \left\{\sigma^0, -\boldsymbol{\sigma}\right\} \qquad \sigma^i = \text{Pauli matrix}$$

Raise/lower spinor index $\xi'_\alpha = A_\alpha{}^\beta \xi_\beta \longrightarrow \xi'^\alpha = \mathsf{C}^{-1\ \alpha\beta} A_\beta{}^\gamma \mathsf{C}_{\gamma\delta}\xi^\delta$

Vector representation of $O(3,1)$ by $SL(2,C)$

$$X = x^0\sigma^0 + x^1\sigma^1 + x^2\sigma^2 + x^3\sigma^3 \longrightarrow X' = AXA^\dagger$$

Conserves $\det X = \det X' = x^\mu x_\mu$ because $\det A = 1$

Little group

$\widehat{A}_\pi \in SL(2,C)$ associated with $\Lambda_\pi \in SO(3,1)$ is stabilizer of π

$\mathcal{L}(\pi) = \left\{ \widehat{A}_\pi \in SL(2,C) \;\middle|\; \pi' = \widehat{A}_\pi \pi \widehat{A}_\pi^\dagger = \pi \right\}$

Fix standard momentum $\mathring{\pi}$ with known $\mathcal{L}(\mathring{\pi})$

Wigner operator

$\alpha(\pi) \in SL(2,C)$ generated by $N_{\mu\nu}$ that takes $\mathring{\pi} \longrightarrow \pi$

$$\pi = \alpha(\pi)\, \mathring{\pi}\, \alpha^\dagger(\pi)$$

Construct $\mathcal{L}(\pi)$ from $\mathcal{L}(\mathring{\pi})$

$$\widehat{A}_{\mathring{\pi}} \in \mathcal{L}(\mathring{\pi}) \longrightarrow \widehat{A}_\pi = \alpha(\pi)\, \widehat{A}_{\mathring{\pi}}\, \alpha^{-1}(\pi) \in \mathcal{L}(\pi)$$

Isomorphism $SL(2,C) \approx \mathcal{L}(\mathring{\pi})$

$$A \in SL(2,C) \longrightarrow A = \alpha(\pi')\, \widehat{A}_{\mathring{\pi}}\, \alpha^\dagger(\pi)$$

Quantum states: unitary representation of Poincaré group

Fix standard $\mathring{\pi}$ and Wigner operator

Standard momentum

Horwitz-Piron-Reuse chose arbitrary timelike vector $\mathring{n} = (1, 0, 0, 0)$

In $SL(2, C)$ representation $\mathring{n} = 1 \cdot \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Associate $\pi^\mu = Mn^\mu \longrightarrow \mathring{\pi} = M\sigma^0$

For $\widehat{A}_{\mathring{\pi}} \in \mathcal{L}(\mathring{\pi})$

$$M\mathring{n} = \widehat{A}_{\mathring{\pi}}\ M\mathring{n}\ \widehat{A}^\dagger_{\mathring{\pi}} \longrightarrow \widehat{A}^\dagger_{\mathring{\pi}} = \widehat{A}^{-1}_{\mathring{\pi}} \longrightarrow \mathcal{L}(\mathring{\pi}) = SU(2)$$

Wigner operator

$\alpha(\pi) = \exp(\boldsymbol{\beta} \cdot \boldsymbol{\sigma}/2) =$ pure boost in direction $\hat{\boldsymbol{\beta}}$

General momentum π

$$\pi = \alpha^\dagger(\pi)\,\mathring{\pi}\,\alpha(\pi) = \alpha(\pi)\,M\sigma^0\alpha(\pi) = M\left[\alpha(\pi)\right]^2\sigma^0 = M\exp(\boldsymbol{\beta} \cdot \boldsymbol{\sigma})\,\sigma^0$$

$$= M\left(\sigma^0 \cosh\beta + \hat{\boldsymbol{\beta}} \cdot \boldsymbol{\sigma} \sinh\beta\right) \qquad \text{rapidity} \quad \beta = \tanh^{-1}|\boldsymbol{\pi}|/\pi^0$$

Quantum states: unitary representation of Poincaré group

Basis quantities for states with spin

Momentum states with spin

Eigenstates $|\pi, p, \sigma\rangle$ of operators P_μ and Π_μ

Pauli–Lubanski pseudovector

Denote $N_\mu = \Pi_\mu / M$

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma} N^{\nu\lambda} N^\sigma \qquad W_\mu N^\mu = 0 \qquad \left[W_\mu, N_\rho\right] = 0$$

Casimir invariant $W^\mu W_\mu \longrightarrow$ spin operator $\frac{1}{2} N^{ij} N_{ij}$ in frame $\mathring{n}$

Unitary representation $U(\Lambda)$ of Lorentz transformation Λ

$$P^\mu \; U(\Lambda) |\pi, p, \sigma\rangle = p'^\mu \; U(\Lambda) |\pi, p, \sigma\rangle = \Lambda^\mu{}_\nu p^\nu \; U(\Lambda) |\pi, p, \sigma\rangle$$

$$\Pi^\mu \; U(\Lambda) |\pi, p, \sigma\rangle = \pi'^\mu \; U(\Lambda) |\pi, p, \sigma\rangle = \Lambda^\mu{}_\nu \pi^\nu \; U(\Lambda) |\pi, p, \sigma\rangle$$

Frame covariance

$$\psi'(\pi', p') = \psi(\pi, p) \longrightarrow \psi'(\pi, p) = \psi(\Lambda^{-1} n, \Lambda^{-1} p)$$

Identity operator for states with spin

$$I = \sum_{\sigma'} \int d\mu\left(p'\right) d\mu\left(\pi'\right) \left|\pi', p', \sigma'\right\rangle \left\langle \pi,' p'\sigma'\right|$$

Matrix element for $U\left(\Lambda\right)$

$$\left\langle \pi', p', \sigma' \right| U\left(\Lambda\right) \left|\pi, p, \sigma\right\rangle = \delta^4\left(p' - \Lambda p\right) \delta^4\left(\pi' - \Lambda\pi\right) \; V_{\sigma'\sigma}\left(\pi, p, \Lambda\right)$$

Unitarity

$$1 = U\left(\Lambda\right) U\left(\Lambda\right)^\dagger \;\longrightarrow\; \sum_{\sigma'} V_{\sigma\sigma'} V^*_{\sigma''\sigma'} = \sum_{\sigma'} V_{\sigma\sigma'} \left(V_{\sigma'\sigma''}\right)^\dagger = \delta_{\sigma\sigma''}$$

Lorentz transformation on spin state

$$U\left(\Lambda\right) \left|\pi, p, \sigma\right\rangle = \sum_{\sigma'} V_{\sigma'\sigma}\left(\pi, p, \Lambda\right) \; \left|\Lambda\pi, \Lambda p, \sigma'\right\rangle$$

Unitary representation of Wigner operator $\alpha(\pi)$

Pure boost constructed from $N^{0i} \longrightarrow$ does not act on p or σ

Define $U(\alpha(\pi))\,|\mathring{\pi}, p, \sigma\rangle = |\pi, p, \sigma\rangle$

Little group

General $A \in SL(2, C)$ associated with $\Lambda \in SO(3,1)$

$V_{\sigma'\sigma}(\pi, p, \Lambda) =$ unitary representation of Λ

Little group element $\widehat{A}_{\mathring{\pi}} = \alpha^{-1}(\pi') A \alpha(\pi) \in \mathcal{L}(\mathring{\pi})$

$$\langle \pi, p', \sigma' | U\left(\widehat{\Lambda}_{\mathring{\pi}}\right) |\mathring{\pi}, p, \sigma\rangle = V_{\sigma''\sigma}(\pi, p, \Lambda)\, \delta^4(p - p')\, \delta^4(\pi - \mathring{\pi})$$

$V_{\sigma'\sigma}(\pi, p, \Lambda) =$ unitary representation of $\widehat{\Lambda}_{\mathring{\pi}} \in \mathcal{L}(\mathring{\pi}) = SU(2)$

Matrix element for $U(\Lambda)$

$$\langle \pi', p', \sigma' | U(\Lambda) |\pi, p, \sigma\rangle = \delta^4(p' - \Lambda p)\, \delta^4(\pi' - \Lambda \pi)\ V_{\sigma'\sigma}(\pi, p, \Lambda)$$

Pure boost of p^μ and π^μ

Rotation of spin indices in hypersurface normal to π^μ

Wavefunction transformation

$$\psi'_\sigma(\pi,p) = \sum_{\sigma'} V_{\sigma'\sigma}(\pi,p,\Lambda)\, \psi_{\sigma'}\left(\Lambda^{-1}\pi, \Lambda^{-1}p\right)$$

$$V_{\sigma'\sigma}(\pi,p,\Lambda) = \text{unitary representation of } \widehat{\Lambda}_{\mathring{\pi}} \in \mathcal{L}(\mathring{\pi})$$

$$\widehat{A}_{\mathring{\pi}} \in SU(2) \;\longrightarrow\; U\left(\widehat{\Lambda}_{\mathring{\pi}}\right) = \widehat{A}_{\mathring{\pi}} = \alpha^{-1}(\pi')\, A\, \alpha(\pi)$$

$$\begin{aligned}\psi'_\sigma(\pi,p) &= \sum_{\sigma'} \left[\alpha^{-1}(\pi) A \alpha(\Lambda^{-1}\pi)\right]_{\sigma'\sigma}\; \psi\left(\Lambda^{-1}\pi, \Lambda^{-1}p\right)_{\sigma'} \\ &= \sum_{\sigma'} \left[\alpha^{-1}(\pi)\; A\right]_{\sigma'\sigma}\; \left[\alpha(\Lambda^{-1}\pi)\psi\left(\Lambda^{-1}\pi, \Lambda^{-1}p\right)\right]_{\sigma'}\end{aligned}$$

Multiply both sides by $\alpha(\pi)$

$$\left[\alpha(\pi)\psi(\pi,p)\right]'_\sigma = \sum_{\sigma'} A_{\sigma',\sigma} \left[\alpha(\Lambda^{-1}\pi)\psi\left(\Lambda^{-1}\pi, \Lambda^{-1}p\right)\right]_{\sigma'}$$

Spinor basis states

$\alpha\psi$ undergoes Lorentz transform as $(\alpha\psi)' = A\,(\alpha\psi)$

Inequivalent representation transforms as $(\underline{\alpha}\phi)' = \underline{A}\,(\underline{\alpha}\phi)$

Quantum mechanics in extended phase space

Bispinors

Horwitz bispinor

$$\Psi(n,x) = \frac{1}{\sqrt{2}} \begin{bmatrix} \alpha\psi(n,x) + \underline{\alpha}\phi(n,x) \\ -\alpha\psi(n,x) + \underline{\alpha}\phi(n,x) \end{bmatrix}$$

γ^μ matrices with Hestenes notation

State transforms as $\Psi'(n,x) = S(\Lambda)\Psi(\Lambda^{-1}n, \Lambda^{-1}x)$

$$S(\Lambda) = 1 - \frac{i}{2}\Sigma^{\mu\nu}\omega_{\mu\nu} \longrightarrow \Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}\gamma^\mu \wedge \gamma^\nu$$

$$K^\mu = \Sigma^{\mu\nu} n_\nu = \frac{i}{2}\gamma^\mu \wedge n \longrightarrow K^\mu n_\mu = \frac{i}{2} n \wedge n = 0$$

Transverse operators

Projection of basis vectors $\gamma_\perp^\mu = \gamma^\mu - n(n \cdot \gamma^\mu)$

$$\Sigma_\perp^{\mu\nu} = \frac{i}{2}\gamma_\perp^\mu \wedge \gamma_\perp^\nu = \Sigma^{\mu\nu} + K^\nu n^\mu - K^\mu n^\nu \qquad \longrightarrow \qquad \Sigma_\perp^{\mu\nu} n_\nu = 0$$

6 independent components of K^μ and $\Sigma_\perp^{\mu\nu}$ satisfy Poincaré Lie algebra

Generate boosts / rotations in spacelike hypersurface transverse to n^μ

Decompose spacetime momentum

$$p_{\|} = \frac{1}{2}(p + npn) \qquad\qquad p_{\perp} = \frac{1}{2}(p - npn)$$

$K_L = p_{\|}$ and $K_T = \gamma^5 p_{\perp}$ Hermitian with respect to standard γ^μ matrices

Free Hamiltonian

$$K_0 = \frac{1}{2M}\left(K_L^2 - K_T^2\right) = p_{\|}^2 + p_{\perp}^2 = \frac{p^2}{2M}$$

Minimal gauge substitution

$$K_L^2 = \left(p_{\|} - eA_{\|}\right)\cdot\left(p_{\|} - eA_{\|}\right) + \left(p_{\|} - eA_{\|}\right)\wedge\left(p_{\|} - eA_{\|}\right) = \left(p_{\|} - eA_{\|}\right)^2$$

$$K_T^2 = \gamma^5\left(p_{\perp} - eA_{\perp}\right)\gamma^5\left(p_{\perp} - eA_{\perp}\right) = -\left(p_{\perp} - eA_{\perp}\right)^2 + e\left(p_{\perp}\wedge A_{\perp}\right)$$

$$p_{\perp}\wedge A_{\perp} = \left(p_\mu A_\nu\right)\gamma_{\perp}^{\mu}\wedge\gamma_{\perp}^{\nu} = -\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\Sigma_{\perp}^{\mu\nu}$$

Horwitz electromagnetic Hamiltonian

$$K = \frac{1}{2M}\left(K_L^2 - K_T^2\right) = \frac{1}{2M}(p - eA)^2 + \frac{e}{2M}F_{\mu\nu}\Sigma_{\perp}^{\mu\nu}$$

Minimal gauge substitution

$$K_L^p = p_\parallel - eA_\parallel \qquad K_T^p = \gamma^5 \left(p_\perp - eA_\perp\right)$$

$$K_L^\pi = \pi_\parallel - e\chi_\parallel \qquad K_T^\pi = \gamma^5 \left(\pi_\perp - e\chi_\perp\right)$$

Extended electromagnetic Hamiltonian

$$K = \frac{1}{2M}\left[\left(K_L^p\right)^2 - \left(K_T^p\right)^2\right] + \frac{1}{2M}\left[\left(K_L^\pi\right)^2 - \left(K_T^\pi\right)^2\right]$$

Commutation relations $\left[p_\mu, A_\nu\right] = i\frac{\partial}{\partial x^\mu}A_\nu \qquad \left[\pi_\mu, \chi_\nu\right] = i\frac{\partial}{\partial n^\mu}\chi_\nu$

$$K = \frac{1}{2M}\left[(p - eA)^2 + (\pi - e\chi)^2\right] + \frac{e}{2M}\left(F_{\mu\nu} + G_{\mu\nu}\right)\Sigma_\perp^{\mu\nu}$$

Field strengths

$$F^{\mu\nu} = \partial A^\nu / \partial x_\mu - \partial A^\mu / \partial x_\nu \qquad G^{\mu\nu} = \partial \chi^\nu / \partial \zeta_\mu - \partial \chi^\mu / \partial \zeta_\nu$$

$H^{\mu\nu} = \partial \chi^\nu / \partial x_\mu - \partial A^\mu / \partial \zeta_\nu$ does not appear

Free particle Stueckelberg-Schrodinger equation

$$i\partial_\tau \Psi(\zeta, x, \tau) = \left(\frac{p^2}{2M} + \frac{\pi^2}{2M}\right)\Psi(\zeta, x, \tau)$$

Plane wave solution

$$\Psi(\zeta, x, \tau) = \begin{bmatrix} \chi^{(1)}(\pi) \\ \chi^{(2)}(\pi) \\ \chi^{(3)}(\pi) \\ \chi^{(4)}(\pi) \end{bmatrix} \exp\left[i\left(p \cdot x + \pi \cdot \zeta - \frac{p^2 + \pi^2}{2M}\tau\right)\right]$$

Constant amplitudes $\chi^{(\sigma)}(\pi) \longrightarrow \mathcal{N}\ \psi^{(\sigma)}$ in frame $\pi = \mathring{\pi} = M(1,0,0,0)$

$\mathcal{N}$ is some normalization

$$\psi^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \psi^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \psi^{(4)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Quantum mechanics in extended phase space

Boost to general frame

Boost $\mathring{\pi}^\mu$

$$\mathring{\pi}^\mu \ \longrightarrow \ \pi^\mu = \Lambda\mathring{\pi} = \exp\left(i\beta^k M_{0k}\right)\mathring{\pi} = M\left(\cosh\beta, \sinh\beta\hat{\boldsymbol{\beta}}\right)$$

State transforms as

$$\Psi^{(\sigma)}(\zeta, x, \tau) = S(\Lambda)\Psi^{(\sigma)}(\Lambda^{-1}\zeta, \Lambda^{-1}x) = S(\Lambda)\Psi^{(\sigma)}(\zeta, x, \tau)$$

Phase of plane wave is Lorentz invariant

$$S\left(\Lambda\right) = \exp\left(-i\Sigma^{0k}\beta_k\right) = \begin{bmatrix} \cosh\dfrac{\beta}{2}\,\sigma^0 & \sinh\dfrac{\beta}{2}\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} \\ \sinh\dfrac{\beta}{2}\,\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\sigma} & \cosh\dfrac{\beta}{2}\,\sigma^0 \end{bmatrix}$$

Four independent solutions

$$\Psi^{(\sigma)}(\zeta, x, \tau) = \mathcal{N}\ u^{(\sigma)}\exp\left[i\left(p\cdot x + \pi\cdot\zeta - \frac{p^2+\pi^2}{2M}\tau\right)\right]$$

$$u^{(\sigma)} = S\left(\Lambda\right)\psi^{(\sigma)}$$

Quantum mechanics in extended phase space

Transformed amplitudes

$$u^{(1)} = \begin{bmatrix} \cosh\frac{\beta}{2} \\ 0 \\ \sinh\frac{\beta}{2}\hat{\beta}^3 \\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1 + i\hat{\beta}^2\right) \end{bmatrix} \qquad u^{(2)} = \begin{bmatrix} 0 \\ \cosh\frac{\beta}{2} \\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1 - i\hat{\beta}^2\right) \\ -\sinh\frac{\beta}{2}\hat{\beta}^3 \end{bmatrix}$$

$$u^{(3)} = \begin{bmatrix} \sinh\frac{\beta}{2}\hat{\beta}^3 \\ \sinh\frac{\beta}{2}\left(\hat{\beta}^1 + i\hat{\beta}^2\right) \\ \cosh\frac{\beta}{2} \\ 0 \end{bmatrix} \qquad u^{(4)} = \begin{bmatrix} \sinh\frac{\beta}{2}\left(\hat{\beta}^1 - i\hat{\beta}^2\right) \\ -\sinh\frac{\beta}{2}\hat{\beta}^3 \\ 0 \\ \cosh\frac{\beta}{2} \end{bmatrix}$$

Conjugate amplitudes

$$\bar{u}^{(\sigma)} = \overline{\left[S\left(\Lambda\right)\psi^{(\sigma)}\right]} = \left[\left(C^{-1}S\left(\Lambda\right)C\right)\psi^{(\sigma)}\right]^{\dagger} = \left[S^{-1}\left(\Lambda\right)\psi^{(\sigma)}\right]^{\dagger}$$

$$\overline{\Psi}^{(\sigma)}(\zeta,x,\tau)\Psi^{(\sigma)}(\zeta,x,\tau) = \mathcal{N}^2\,\bar{u}^{(\sigma)}u^{(\sigma)} = \mathcal{N}^2$$

Quantum mechanics in extended phase space

Spin-1/2 state

Pauli–Lubanski pseudovector

$$W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}N^{\nu\lambda}N^\sigma \longrightarrow W_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\Sigma_\perp^{\nu\lambda}N^\sigma = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}\Sigma^{\nu\lambda}N^\sigma$$

$$W_0 = -\frac{1}{2}\begin{bmatrix} \boldsymbol{\sigma}\cdot\boldsymbol{n} & 0 \\ 0 & \boldsymbol{\sigma}\cdot\boldsymbol{n} \end{bmatrix} \qquad W_i = \frac{1}{2}\begin{bmatrix} n^0\delta_{ik}\sigma^k & i\,(\boldsymbol{\sigma}\times\boldsymbol{n})_i \\ i\,(\boldsymbol{\sigma}\times\boldsymbol{n})_i & n^0\delta_{ik}\sigma^k \end{bmatrix}$$

Frame $n = \mathring{n} = (1,0,0,0) \longrightarrow W_0 = 0$ and W_3 diagonal

$$W_3\Psi^{(1)} = +\Psi^{(1)} \qquad W_3\Psi^{(2)} = -\Psi^{(2)} \qquad W_3\Psi^{(3)} = +\Psi^{(3)} \qquad W_3\Psi^{(4)} = -\Psi^{(4)}$$

Total spin

$$W^\mu W_\mu = \frac{1}{2}\Sigma^{\nu\lambda}\Sigma_{\nu\lambda}n^\sigma N_\sigma - \Sigma_{\mu\sigma}\Sigma^{\nu\sigma}N^\mu N_\nu = \frac{1}{2}\Sigma^{\nu\lambda}\Sigma_{\nu\lambda}$$

Independent of N^μ and commutes with all other generators

$$-W^\mu W_\mu = -W_0^2 - \eta^{ii'}W_iW_{i'} = \frac{3}{4}\begin{bmatrix} \sigma^0 & 0 \\ 0 & \sigma^0 \end{bmatrix} = \frac{1}{2}\left(\frac{1}{2}+1\right)$$

Bispinor is spin-1/2 state

Quantum mechanics in extended phase space

Vector spin operator

Orthonormal basis $\boldsymbol{e}^i$ for spacelike hypersurface

$$W_\mu n^\mu = 0 \;\longrightarrow\; W = J_k\ \boldsymbol{e}^k \;\longrightarrow\; W^\mu = W \cdot \gamma^\mu = J_k\ \left(\boldsymbol{e}^k\right)^\mu$$

Components J_k are vector spin operator

Total spin is $-\eta_{\mu\nu} W^\mu W^\nu = -\eta^{kk'} J_k J_{k'} = \boldsymbol{J}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)$

n = boost of $\mathring{n}$ along 3-axis $\longrightarrow$ orthonormal basis $\boldsymbol{e}^\mu$

$$\boldsymbol{e}^0 = n = n^{(0)}\gamma^0 - n^{(3)}\gamma^{(3)} \qquad \boldsymbol{e}^1 = \gamma^1 \qquad \boldsymbol{e}^2 = \gamma^2 \qquad \boldsymbol{e}^3 = n^{(0)}\gamma^3 - n^{(3)}\gamma^0$$

$$n^{(0)} = \sqrt{1 + n_{(3)}^2}$$

Vector components $\quad J^k = W \cdot \boldsymbol{e}^k = \left(W_0\gamma^0 + W_i\gamma^i\right)\cdot \boldsymbol{e}^k$

$$J_1 = \frac{1}{2}\begin{bmatrix} n^0\sigma^1 & in^3\sigma^2 \\ in^3\sigma^2 & n^0\sigma^1 \end{bmatrix} \qquad J_2 = \frac{1}{2}\begin{bmatrix} n^0\sigma^2 & -in^3\sigma^1 \\ -in^3\sigma^1 & n^0\sigma^2 \end{bmatrix}$$

$$J_3 = \frac{1}{2}\begin{bmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{bmatrix}$$

Plane wave state

$$\Psi^{(\sigma)}(\zeta, x, \tau) = \varphi(\zeta, x, \tau)\, u^{(\sigma)}(\pi)$$

$$\varphi(\zeta, x, \tau) = \mathcal{N} \exp\left[i\left(p \cdot x + \pi \cdot \zeta - \frac{p^2 + \pi^2}{2M}\tau\right)\right]$$

Singlet state

$$\Psi^{(0)}(\zeta_1, \zeta_2, x_1, x_2, \tau) = \varphi(\zeta_1, \zeta_2, x_1, x_2, \tau)\, u^{(0)}(\pi)$$

Spacetime part symmetric under $(\zeta_1, x_1) \leftrightarrow (\zeta_2, x_2)$

$$\varphi(\zeta_1, \zeta_2, x_1, x_2, \tau) = \frac{1}{\sqrt{2}}\left[\varphi_1(\zeta_1, x_1, \tau)\, \varphi_2(\zeta_2, x_2, \tau) + \varphi_2(\zeta_1, x_1, \tau)\, \varphi_1(\zeta_2, x_2, \tau)\right]$$

Spin part requires $J_3\, u^{(\sigma_1)} + J_3\, u^{(\sigma_2)} = 0$ and antisymmetry under $\sigma_1 \leftrightarrow \sigma_2$

$$u^{(0)}(\pi) = \frac{1}{\sqrt{2}}\left[u_1^{(\sigma_1)}(\pi)\, u_2^{(\sigma_2)}(\pi) - u_1^{(\sigma_2)}(\pi)\, u_2^{(\sigma_1)}(\pi)\right]$$

One-particle states transform under same representation of $SL(2, C)$

Parity permits singlets from pairs $\{u^1, u^2\}$ and $\{u^3, u^4\}$

First order perturbation in constant field

First order Hamiltonian

To first order in e

$$K = \frac{1}{2M}\left(p^2 + \pi^2\right) - \frac{e}{M}\left(A \cdot p + \chi \cdot \pi\right) + \frac{e}{2M}\left(F_{\mu\nu} + G_{\mu\nu}\right)\Sigma_{\perp}^{\mu\nu}$$

Lorenz condition $\qquad \dfrac{\partial}{\partial x^\mu} A^\mu = 0 \qquad \dfrac{\partial}{\partial \zeta^\mu}\chi^\mu = 0$

Constant field strengths $F_{\mu\nu} = G_{\mu\nu}$

$$A^\mu = -\frac{1}{2}F^{\mu\nu}x_\nu \qquad \chi^\mu = -\frac{1}{2}F^{\mu\nu}\zeta_\nu$$

$$-\frac{e}{M}\left(A \cdot p + \chi \cdot \pi\right) = \frac{e}{2M}F^{\mu\nu}\left(x_\mu p_\nu + \zeta_\mu \pi_\nu\right)$$

Perturbed Hamiltonian

$$K = K_0 - \frac{e}{4M}F_{\mu\nu}\left(L^{\mu\nu} + N^{\mu\nu}\right) + \frac{e}{M}F_{\mu\nu}\Sigma_{\perp}^{\mu\nu} \quad \text{by antisymmetry of } F^{\mu\nu}$$

$L^{\mu\nu}$ and $N^{\mu\nu}$ generate Lorentz transformations on extended spacetime

First order perturbation in constant field

Magnetic field

Pure magnetic field

$$F_{0\nu} = 0 \qquad F_{ij} = \varepsilon_{ijk} B^k$$

Interaction terms

$$\frac{1}{2} F_{\mu\nu} \left(L^{\mu\nu} + N^{\mu\nu} \right) = B \cdot J \qquad F_{\mu\nu} \Sigma_{\perp}^{\mu\nu} = B \cdot \mathcal{J}$$

$$J_k = \frac{1}{2} \epsilon_{ijk} M^{ij} = (\boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{\zeta} \times \boldsymbol{\pi})_k \qquad \mathcal{J}^i = \epsilon^i{}_{jk} \Sigma_{\perp}^{jk}$$

Orbital angular momentum in extended spacetime and spin

J_k vanishes for plane wave solution $\longrightarrow$ $K_{spin} = \dfrac{e}{M} B_k \mathcal{J}^k$

In frame defined by pure boost along 3-axis $\mathring{\pi} \longrightarrow \pi = M \left(n^{(0)}, 0, 0, n^{(3)} \right)$

$$K_{spin} = \frac{e}{2M} \begin{bmatrix} B^3 \sigma^3 + n^2_{(0)} \left(B^1 \sigma^1 + B^2 \sigma^2 \right) & i n^{(0)} n^{(3)} \left(B^1 \sigma^2 - B^2 \sigma^1 \right) \\ i n^{(0)} n^{(3)} \left(B^1 \sigma^2 - B^2 \sigma^1 \right) & B^3 \sigma^3 + n^2_{(0)} \left(B^1 \sigma^1 + B^2 \sigma^2 \right) \end{bmatrix}$$

First order perturbation in constant field

Magnetic field along 3-axis

Magnetic field $\boldsymbol{B} = (0, 0, B^3)$ along 3-axis

$$K_{spin}\ \Psi^{(\sigma)} = \begin{cases} +\left(eB^3/2M\right)\Psi^{(\sigma)} & , \quad \sigma = 1,3 \\ -\left(eB^3/2M\right)\Psi^{(\sigma)} & , \quad \sigma = 2,4 \end{cases}$$

$$\langle p, \pi, \sigma' |\, K\, | p, \pi, \sigma\rangle = \begin{cases} \mathcal{N}^2 \delta^{\sigma\sigma'} \left[\dfrac{p^2 + \pi^2}{2M} + \dfrac{eB^3}{2M}\Psi^{(\sigma)}\right] & , \quad \sigma = 1,3 \\ \mathcal{N}^2 \delta^{\sigma\sigma'} \left[\dfrac{p^2 + \pi^2}{2M} - \dfrac{eB^3}{2M}\Psi^{(\sigma)}\right] & , \quad \sigma = 2,4 \end{cases}$$

Total mass eigenvalues of {1,2} and {3,4} singlets conserved

$$u^{(1,2)}(\pi) = \frac{1}{\sqrt{2}}\left[u_1^{(1)}(\pi)\, u_2^{(2)}(\pi) - u_1^{(2)}(\pi)\, u_2^{(1)}(\pi)\right]$$

$$u^{(3,4)}(\pi) = \frac{1}{\sqrt{2}}\left[u_1^{(3)}(\pi)\, u_2^{(4)}(\pi) - u_1^{(4)}(\pi)\, u_2^{(3)}(\pi)\right]$$

First order perturbation in constant field

Magnetic field along 1-axis

Magnetic field $\boldsymbol{B} = \left(B^1, 0, 0\right)$ along 1-axis

$$K_{spin} = \frac{eB^1}{2M} \left[\begin{array}{cc} n_{(0)}^2 \sigma^1 & n^{(0)} n^{(3)} i\sigma^2 \\ n^{(0)} n^{(3)} i\sigma^2 & n_{(0)}^2 \sigma^1 \end{array} \right]$$

$$\boxed{n = (\cosh\beta, 0, 0, \sinh\beta)}$$

Real but off-diagonal

Produces transition in spin states

No shift in π^μ that could disrupt the singlet

Non-zero transition amplitudes

$$\langle p, \pi, 2 | K_{spin} | p, \pi, 1 \rangle = \langle p, \pi, 1 | K_{spin} | p, \pi, 2 \rangle = \left(eB^1/2M\right) \cosh\beta$$

$$\langle p, \pi, 4 | K_{spin} | p, \pi, 3 \rangle = \langle p, \pi, 3 | K_{spin} | p, \pi, 4 \rangle = \left(eB^1/2M\right) \cosh\beta$$

Transition

$$u^{(1,2)}(\pi) \longrightarrow \frac{1}{\sqrt{2}} \left[u_1^{(2)}(\pi) \, u_2^{(1)}(\pi) - u_1^{(1)}(\pi) \, u_2^{(2)}(\pi) \right] = -u^{(1,2)}(\pi)$$

Equivalent to exchange of particles

First order perturbation in constant field

Electric field

Pure electric field

$$F_{0i} = E_i \qquad F_{ij} = 0$$

Interaction terms

$$\frac{e}{4M} F_{\mu\nu} M^{\nu\mu} = -\frac{e}{2M} E_i \left(L^{0i} + N^{0i} \right) \qquad \frac{e}{M} F_{\mu\nu} \Sigma_{\perp}^{\mu\nu} = \frac{2e}{M} E_i \Sigma_{\perp}^{0i}$$

$$L^{0i} = x^0 p^i - x^i p^0 \qquad L_n^{0i} = \zeta^0 \pi^i - \zeta^i \pi^0$$

Expectation values of extended spacetime boost operators vanish

Boost operator in spin space

$$\Sigma_{\perp}^{0i} = \Sigma^{0i} - \Sigma^{0i} n_0 n^0 - \Sigma^{0j} n_j n^i + \Sigma^{ij} n_j n^0$$

$$K_{spin} = \frac{2e}{M} E_i \, \Sigma_{\perp}^{0i} = \frac{e}{M} \left[\begin{array}{cc} 0 & T(\boldsymbol{\sigma}) \\ T(\boldsymbol{\sigma}) & 0 \end{array} \right]$$

$$T(\boldsymbol{\sigma}) = -i n_{(3)}^2 \left(E_1 \sigma^1 + E_2 \sigma^2 \right) - n^{(0)} n^{(3)} \left(E^1 \sigma^2 - E^2 \sigma^1 \right)$$

First order perturbation in constant field

Perturbations from anti-Hermitian and non-compact boost generators

Electric field $E = \left(0, 0, E^3\right)$ parallel to boost of $\mathring{\pi} \longrightarrow \pi = M\left(n^{(0)}, 0, 0, n^{(3)}\right)$

$$T(\sigma) = -in^2_{(3)}\left(E_1\sigma^1 + E_2\sigma^2\right) - n^{(0)}n^{(3)}\left(E^1\sigma^2 - E^2\sigma^1\right) = 0$$

Electric field $E = \left(E^1, 0, 0\right)$

$$K_{spin} = i\frac{eE^1}{M}n^{(3)}\begin{bmatrix} 0 & n^{(3)}\sigma^1 + n^{(0)}i\sigma^2 \\ n^{(3)}\sigma^1 + n^{(0)}i\sigma^2 & 0 \end{bmatrix}$$

Non-zero transition amplitudes

$$\langle 2| K_{spin} |1\rangle = i\frac{eE^3}{M}e^{-\beta}\sinh\beta \qquad \langle 1| K_{spin} |2\rangle = i\frac{eE^3}{M}e^{\beta}\sinh\beta$$

$$\langle 4| K_{spin} |1\rangle = i\frac{eE^3}{M}e^{-\beta}\cosh\beta \qquad \langle 1| K_{spin} |4\rangle = -i\frac{eE^3}{M}e^{\beta}\cosh\beta$$

$$\langle 2| K_{spin} |3\rangle = i\frac{eE^3}{M}e^{-\beta}\cosh\beta \qquad \langle 3| K_{spin} |2\rangle = -i\frac{eE^3}{M}e^{\beta}\cosh\beta$$

$$\langle 4| K_{spin} |3\rangle = i\frac{eE^3}{M}e^{-\beta}\sinh\beta \qquad \langle 3| K_{spin} |4\rangle = i\frac{eE^3}{M}e^{\beta}\sinh\beta$$

First order perturbation in constant field

Perturbative transitions in electric field normal to π

Allowed transitions for $u^{(1)}$

$$\langle 2|\, K_{spin}\, |1\rangle = i\frac{eE^3}{M} e^{-\beta} \sinh\beta$$

$$\langle 1|\, K_{spin}\, |2\rangle = i\frac{eE^3}{M} e^{\beta} \sinh\beta = -i\frac{eE^3}{M} e^{\beta} \sinh(-\beta) = \langle 2|\, K_{spin}\, |1\rangle^*_{\beta \to -\beta}$$

$$\langle 4|\, K_{spin}\, |1\rangle = i\frac{eE^3}{M} e^{-\beta} \cosh\beta$$

$$\langle 1|\, K_{spin}\, |4\rangle = -i\frac{eE^3}{M} e^{\beta} \cosh\beta = \langle 4|\, K_{spin}\, |1\rangle^*_{\beta \to -\beta}$$

Generally for $1 \leftrightarrow 2,4$ and $3 \leftrightarrow 2,4$

$$\langle \sigma|\, K_{spin}\, |\sigma'\rangle = \langle \sigma'|\, K_{spin}\, |\sigma\rangle^*_{\beta \to -\beta}$$

$\beta \longrightarrow -\beta \implies$ parity transformation of π

$$\pi = M\left(\cosh\beta, 0, 0, \sinh\beta\right) \xrightarrow[\beta \to -\beta]{} \mathcal{P}[\pi] = M\left(\cosh\beta, 0, 0, -\sinh\beta\right)$$

First order perturbation in constant field

Parity transitions in electric field

$\beta \rightarrow -\beta$ for spin states

$$u^{(1)} = (\cosh\beta/2, 0, \sinh\beta/2, 0) \xrightarrow[\beta \rightarrow -\beta]{} \mathcal{P}\left[u^{(1)}\right]$$

$$u^{(2)} = (0, \cosh\beta/2, 0, -\sinh\beta/2) \xrightarrow[\beta \rightarrow -\beta]{} \mathcal{P}\left[u^{(2)}\right]$$

$$u^{(3)} = (\sinh\beta/2, 0, \cosh\beta/2, 0) \xrightarrow[\beta \rightarrow -\beta]{} -\mathcal{P}\left[u^{(3)}\right]$$

$$u^{(4)} = (0, -\sinh\beta/2, 0, \cosh\beta/2) \xrightarrow[\beta \rightarrow -\beta]{} -\mathcal{P}\left[u^{(4)}\right]$$

$\left\{u^{(1)}, u^{(2)}\right\}$ and $\left\{u^{(3)}, u^{(4)}\right\}$ transform under inequivalent representations of $SL(2,C)$

Possible transitions induced by perturbation on singlet state $u^{(1,2)}$

$$u^{(1,2)} \longrightarrow u^{(2,1)} \qquad u^{(1,2)} \longrightarrow u^{(4,3)}$$

Pairs belong to same Hilbert space — mutually coherent

$$u^{(1,2)} \longrightarrow u^{(2,3)} \qquad u^{(1,2)} \longrightarrow u^{(4,1)}$$

Pairs belong to different Hilbert spaces — not mutually coherent

Perturbation may disrupt coherence of singlet state