Fluctuations of Quantum Fields & Thermodynamics of Spacetime

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Based on H. T. Cho, J. T. Hsiang and B. L. Hu, *Quantum Capacity and Vacuum Compressibility of Spacetimes Thermal Fields*, *Universe* 8, 291 (2022).
Yucun Xie, J. T. Hsiang and B. L. Hu, *Dynamical Vacuum Compressibility of Space,* Phys. Rev. D109, 065027 (2024)
J. T. Hsiang, Yucun Xie and B. L. Hu, *Heat Capacity and Quantum Compressibility of Dynamical Spacetimes with Thermal Particle Creation*, [arXiv:2405.00360](2024)

1. What is **Quantum Gravity**? Q-C vs m-M
2. Why are we interested in **Quantum Thermodynamics** (QTD)
3. Quantum Thermodynamics of Spacetime: From Quantum Fields in Curved Spacetime to **Semiclassical Gravity**
4. Energy Density Fluctuations comparable to the mean: Alarm! (TD stability considerations): interesting physics?
5. **Fluctuations**. Noise Kernel in **Stochastic Gravity**
6. **Heat Capacity** and **Quantum Compressibility** of Spacetime
7. **Static Spaces**: Einstein cylinder, S^2, S^3 (*Universe 2022*)
8. **Dynamical Spacetimes**: Vacuum Compressibility (*PRD2024*)
9. **Cosmology**: exponential expansion -> thermal particle production

Quantum Gravity

- **In agreement: QG: theories for the microscopic structure of spacetime.** Same noble quest, but the
- **Approaches & Methods toward this lofty goal differ widely.**

- **Conventional approach** (from the 50s on) as practiced in the GR community: Quantize the **metric g** or **connection Γ** form.

One crucial question

remains unanswered, or not directly addressed

- We know **General Relativity** works very well in the description of the **large scale structure** and dynamics of spacetime.
- Metrics and connections are **macroscopic variables.**

Why do you believe that quantizing these variables will reveal the microscopic structures?

EM field, yes, Helium 4, yes.

But is this a universal paradigm? No.

Quantization of a collective variable need **not** lead to **micro-structure**

- There is an abundance of examples in condensed matter physics, which deals mainly with the dynamics of **collective variables of atomic interactions**: phonons, rotons, plasmons, excitons, many `on's:
- One can quantize sound, and study the interaction of phonons with electrons, etc.
- **These `on's are all quantum entities**, but
- **Quantum =/=> microscopic**.

A crucial question to ask, more important than quantization

Are metric and connection forms

- **fundamental**: depicting the microscopic constituents at the Planck scale? or
- **collective** variables constructed from them?

Is general relativity an effective theory valid only at large scales / low energies? Like hydrodynamics with regard to molecular dynamics

General Relativity is Geometro-**hydro**-dynamics

- **My view:**

GR as Hydro (1996), Can spacetime be a condensate? (2005)
Stochastic gravity as mesoscopic physics (1994)
Cosmology as `condensed matter' physics (1988)

- In reference to Quantum Gravity, General Relativity is of the nature of a **hydrodynamic theory,** valid only in the long wavelength, low energy limits of the microscopic theories at the Planck scale =>

- Geometry, manifold structures are derived properties

- Spacetime is an **emergent entity,** so are its symmetries

- **(g, Γ) are collective variables:** Quantizing them gives phonons, not atoms. In fact, sound ceases to exist at the atomic scale, however quantum you make it.

Similar view, **Volovik**: He3 ; **Wen**: tensor network, fermions, light are emergent.
Gravity & Thermodynamics: **Jacobson:** Einstein equation of state (1995)

Quantum-Classical vs micro-Macro

Quantum → decoherence, robustness, stability → **Classical**

← **Traditional effort:** quantizing the metric or connection forms

Quantum Gravity (Strings, Loops, Simplices, Causets – micro const.)

micro	**fluctuations**

coarse-|-graining | ***MESO*** | **kinetic theory**

v *Emergent spacetime* | **hydrodynamics**

MACRO | ***General Relativity***

Issues: Coherence, Correlations, Fluctuations, **Stochasticity**; Collectivity, Variability, Nonlinearity, **Nonlocality**

Besides Quantum to Classical, perhaps more important to first identify the correct micro or meso variables before the consideration of quantization. micro to Macro.

Top Down: **Posit the micro-variables**

- Strings
- Loops
- Causal sets
- Simplices
- Asymptotic safety (RG)
- Group field theory …

Need to show how the nice manifold and causal structure of spacetime emerge. **Emergent Gravity**

Bottom-Up: Quantum Gravity

Start from low energy theory:

General Relativity as

Geometro-Hydrodynamics

- Gravity/GR is a classical theory which makes sense only at the macro scale. Like phonons, but not atoms,

- Metric and connection forms are collective, not "fundamental", variables. Geometry known in the classical context. (meaning of quantum geometry?)

- GR is an effective theory, the low energy, long wave-length limit of some as yet unknown (unconfirmed) theory (many to one) for the micro structure of spacetime and matter (def of quantum gravity)

- Manifold and metric structure, together with the symmetry of physical laws defined thereby (e.g., Lorentz and gauge invariance), are emergent.

Going **from macro looking for micro** structure is always difficult, if not impossible.
BUT not hopeless.. that is how physics has progressed through centuries!

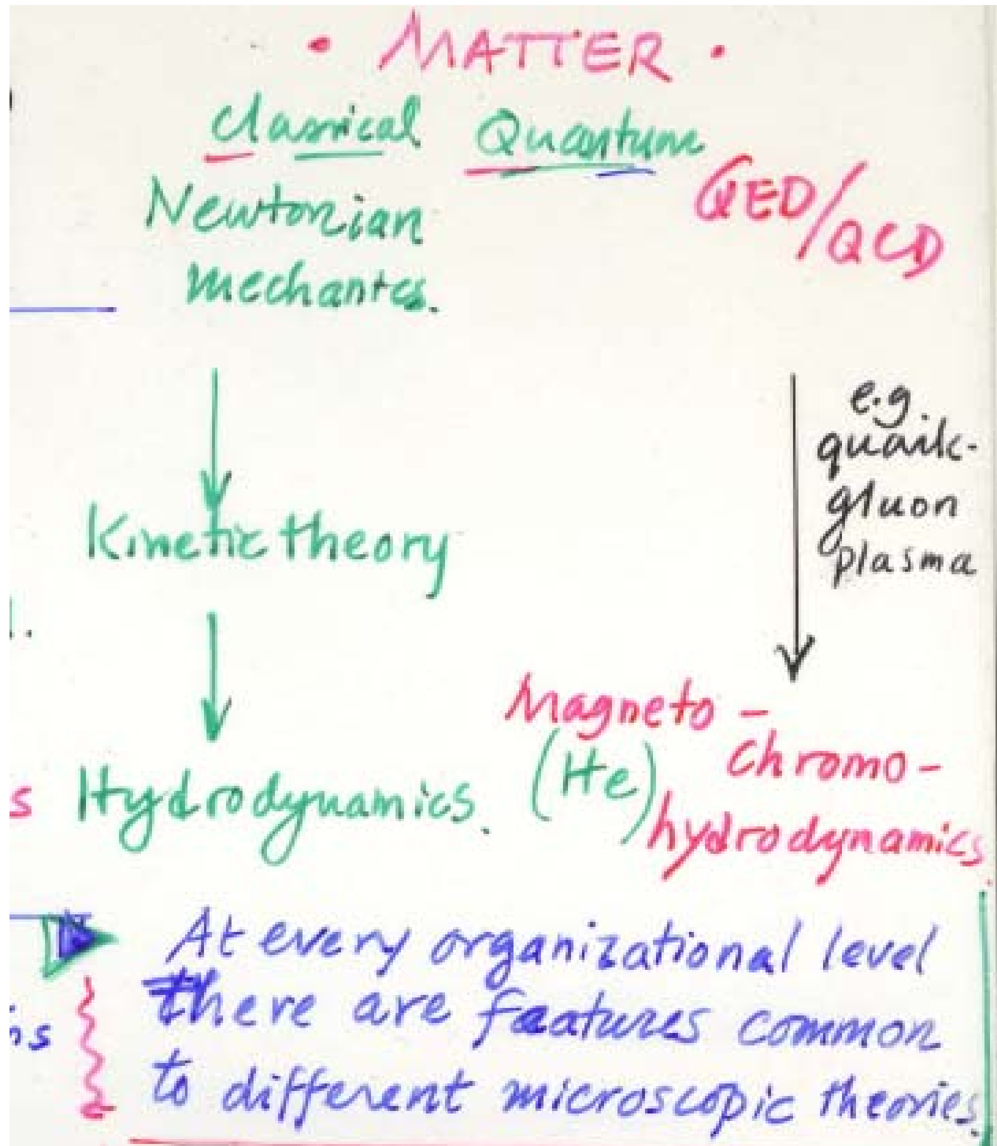
Common Features of Macroscopic Phenomena:

Micro

There are commonalities in the MACroscopic collective behavior of different MICroscopic constituents

Macro

Separate the common features to pinpoint the particulars



Bottom-up: Macro to Micro

We choose to rely on:

A. Topological structures:

More resilient to evolutionary or environmental changes. Many excellent work on topological features in manifolds (wormholes)

Wen, Cirac et al: tensor network, string-nets, emergent spacetime

B. Noise-fluctuations: Fluctuations can reveal some sub-structural contents and behavior (**critical phenomena**).

Information contained in remnants or leftovers.
Reconstruction from corrupted and degraded information

May get some glimpses of the nature of micro-structure.

Deciphering microstructure's generic properties from

Noise and Fluctuation Phenomena

A modest way to move from BOTTOM UP is:

Stochastic Gravity

Main advantage: Minimize speculative assumptions

A natural extension of well known and tested theories:

- Quantum field theory in curved spacetime (e.g., Hawking effect)
- Semiclassical gravity (e.g.,inflationary cosmology)

Stochastic Gravity

- Stochastic Gravity is an upgrade of Semiclassical Gravity (SCG), adding on the considerations of noise and fluctuations of the stress energy tensor of matter fields.

- Semiclassical Gravity involves finding the self consistent solutions to the semiclassical Einstein equation for spacetime dynamics sourced by the expectation values of the stress energy tensor

Semiclassical Gravity

Semiclassical Einstein Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa\langle\hat{T}_{\mu\nu}\rangle_q + \kappa(T_{\mu\nu})_c$$

$\tilde{G}_{\mu\nu}$ is the Einstein tensor (plus covariant terms associated with the renormalization of the quantum field)

$\kappa = 8\pi G_N$ and G_N is Newton's constant

Free massive scalar field

$$(\Box - m^2 - \xi R)\hat{\phi} = 0.$$

$\hat{T}_{\mu\nu}$ is the stress-energy tensor operator

$\langle\,\rangle_q$ denotes the expectation value

Stochastic Gravity

Einstein- Langevin Equation (schematically):

$$\tilde{G}_{\mu\nu}(g_{\alpha\beta}) = \kappa \left(T^{\rm c}_{\mu\nu} + T^{\rm qs}_{\mu\nu} \right)$$

$T^{c}_{\mu\nu}$ is due to classical matter or fields

$$T^{\rm qs}_{\mu\nu} \equiv \langle \hat{T}_{\mu\nu} \rangle_{\rm q} + T^{\rm s}_{\mu\nu}$$

$T^{\rm qs}_{\mu\nu}$ is a new stochastic term
related to the quantum fluctuations of $T_{\mu\nu}$

Einstein-Langevin Equation

- Consider a weak gravitational perturbation h off a background g $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu},$ The ELE is given by (The ELE is Gauge invariant)

$$G_{ab}[g+h] + \Lambda(g_{ab}+h_{ab}) - 2(\alpha A_{ab} + \beta B_{ab})[g+h] = 8\pi G\left(\langle \hat{T}^{R}_{ab}[g+h]\rangle + \xi_{ab}[g]\right).$$

- Nonlocal dissipation and colored noise

  Nonlocality manifests with **stochasticity**

  because this is an open system

Noise Kernel

A physical observable that describes these fluctuations to the lowest order is the noise kernel which is the vacuum expectation value of the two-point correlation function of the stress-energy operator

$$N_{abcd}[g; x, y) = \frac{1}{2}\langle\{\hat{t}_{ab}[g; x), \hat{t}_{cd}[g; y)\}\rangle,$$

$$\hat{t}_{ab}[g; x) \equiv \hat{T}_{ab}[g; x) - \langle\hat{T}_{ab}[g; x)\rangle.$$

NOISE KERNEL

- Exp Value of 2-point correlations of stress tensor: bitensor
- Noise kernel measures quantum flucts of stress tensor

$$N_{abcd}(x,y) = \frac{1}{2}\langle\{\hat{t}_{ab}(x), \hat{t}_{cd}(y)\}\rangle$$

$$\hat{t}_{ab} \equiv \hat{T}_{ab} - \langle\hat{T}_{ab}\rangle\hat{I}$$

It can be represented by (shown via influence functional to be equivalent to) a classical stochastic tensor source $\xi_{ab}[g]$

$$\langle\xi_{ab}\rangle_s = 0 \qquad \langle\xi_{ab}(x)\xi_{cd}(y)\rangle_s = N_{abcd}(x,y)$$

- **Symmetric, traceless** (for conformal field), **divergenceless**

Fluctuations in Quantum Field: Particle Creation

Stochastic Gravity: Physical Meaning of **Noise Kernel.**

Spacetime Thermodynamics: Quantum Thermodynamics

applied to Spacetime.

Role of **Fluctuations in Thermodynamics**

Quantum Capacity & Vacuum Compressibility

Quantum Thermodynamics of Spacetime

I. Heat Capacity and Quantum Compressibility of Static Spacetimes

II. Vacuum Compressibility of Dynamical Spacetimes

III. Heat Capacity of Dynamical Spacetimes with Thermal Particle Creation

(80's): Vacuum Viscosity

-- Zeldovich 1970 JETPLett used this concept to explain particle creation.
Actually it describes its backreaction effects on the dynamics of spacetime

We (with Parker and Hartle) worked this out 1977-1980 rigorously using QFTCST

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(surprisingly) cited by D Jou, J Casas-Vhzquez and G Lebon, Extended irreversible thermodynamics, Rep. Prog. Phys. 51 (1988) 1105-1179 ; W Zimdahl, Bulk viscous cosmology, PRD53, 5483 (1996) de Haro et al, Finite-time cosmological singularities and the possible fate of the Universe, Phys. Rep. 1034 (2023) 1–114

VACUUM VISCOSITY DESCRIPTION OF QUANTUM PROCESSES IN THE EARLY UNIVERSE *

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Quantum dissipative processes in the early universe are discussed in terms of relativistic imperfect fluid formulation. Viscosity functions of the vacuum due to the trace anomaly and particle production of quantized scalar fields in an isotropically expanding spacetime are derived.

I. Heat Capacity and Thermal Compressibility of Static Spacetimes
II. Vacuum Compressibility of Dynamical Spacetimes
III. Heat Capacity of Dynamical Spacetimes with Thermal Particle Creation

In the Lorentzian sector, if we use the MTW signature convention $(-1, 1, \ldots)$, then the Euclidean sector, has signature $(1, 1, \ldots)$. We consider metrics of the form

$$g_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & h_{ij} \end{pmatrix}, \tag{1}$$

where h_{ij} is the metric for the spatial section. We denote the time and spatial variables by $x = (\tau, \mathbf{x})$, the invariant spatial volume form by $d\mathbf{x}$, and the spatial manifold by Σ (thus $\mathcal{M} = S^1 \times \Sigma$).

The wave operator for the Klein-Gordon field is given by

$$H = -\Box + \xi R + m^2 = -\frac{\partial^2}{\partial \tau^2} - {}^{\Sigma}\Delta + \xi R + m^2. \tag{2}$$

Let $u_n(\mathbf{x})$ be the eigenfunctions of ${}^{\Sigma}\Delta$: ${}^{\Sigma}\Delta u_n(\mathbf{x}) = -\kappa_n^2 u_n(\mathbf{x})$, where n denotes the (collective) quantum numbers for the spatial part of the spectrum. We assume the $u_n(\mathbf{x})$ are orthonormal. In thermal field theory the Euclidean time is made periodic with a period of $\beta = 1/T$, where T is the temperature. To calculate the Casimir energy density in the next example, this periodic imaginary time is switched back to a circle in real space, giving the topology of a cylinder.

The eigenfunctions are thus given by

$$\phi_{k_0,n}(x) = \frac{e^{-ik_0\tau}}{\sqrt{\beta}} u_n(\mathbf{x}), \quad k_0 = \frac{2\pi n_0}{\beta}, \quad n_0 = 0, \pm 1, \pm 2, \ldots \tag{3}$$

and the eigenvalues by,

$$\lambda_{k_0,n} = k_0^2 + \kappa_n^2 + \xi R + m^2$$

The stress-energy tensor is defined as

$$T_{ab}[\psi, \phi](x) \equiv \frac{2}{\sqrt{g(x)}} \int \sqrt{g(x')}\psi(x') \left(\frac{\delta H_{x'} \phi(x')}{\delta g^{ab}(x)} \right) dx'$$
$$= -2\nabla_a \psi \nabla_b \phi + g_{ab} \left(\nabla_c \psi \nabla^c \phi + \psi \nabla_c \nabla^c \phi \right) + 2\xi \psi \phi R_{ab}. \tag{5}$$

Phillips and Hu, PRD55, 6103 (1997)

II. STRESS ENERGY TWO-POINT FUNCTION

We start by considering the generating functional (or partition function) of a scalar field ϕ in the Euclidean section Σ of a spacetime manifold $\mathcal{M}$,

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} = \langle 0,\text{out}|0,\text{in}\rangle \tag{2.1}$$

and its functional derivatives with respect to the metric:

In terms of the effective action W=ln Z,

$$\langle T_{ab}\rangle = \frac{\langle 0,\text{out}|T_{ab}|0,\text{in}\rangle}{\langle 0,\text{out}|0,\text{in}\rangle} = -\frac{2}{\sqrt{g(x)}} \frac{\delta W}{\delta g^{ab}(x)}.$$

$$W = -\frac{1}{2}\ln\det(H/\mu) = -\frac{1}{2}\text{Tr}\ln\frac{H}{\mu} = -\frac{1}{2}\sum_n \ln\frac{\lambda_n}{\mu},$$

based on (3.13) PH97

$$T_{ab}[\phi_n(x), \phi_n'^*(x)] \equiv -\frac{2}{\sqrt{g(x)}} \left\langle n' \left| \frac{\delta H}{\delta g_{ab}(x)} \right| n \right\rangle = -\frac{2}{\sqrt{g(x)}} \int d\mathbf{z}\, \phi_n'^*(z) \frac{\delta H}{\delta g_{ab}(x)} \phi_n(z),$$

$$
\Delta T^2_{abcd}(x,y) \equiv \langle T_{ab}(x) T_{cd}(y)\rangle - \langle T_{ab}(x)\rangle\langle T_{cd}(y)\rangle
$$
$$
= \frac{4}{\sqrt{g(x)g(y)}} \frac{\delta^2 W}{\delta g^{ab}(x)\, \delta g^{cd}(y)}.
$$

The energy density fluctuations have been calculated by a number of authors [34]. A dimensionless measure of fluctuations is often used, defined as

$$
\Delta = \left| 1 - \frac{\langle\rho\rangle^2}{\langle\rho^2\rangle} \right| = \left| \frac{\Delta\rho^2}{\Delta\rho^2 + \langle\rho\rangle^2} \right| \tag{6}
$$

For Minkowski space Phillips and Hu [28] calculated this using a smearing field ,

$$
\Delta_{\text{Minkowski}}(d) = \frac{1+d}{1+3\,d} \tag{7}
$$

which has the particular values

d	1	3	5	∞
$\Delta_{\text{Minkowski}}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{3}{8}$	$\frac{1}{3}$

2.1 Casimir Space

The Casimir topology is obtained from a flat space (with d spatial dimensions, i.e., $R^1 \times R^d$) by imposing periodicity L in one of its spatial dimensions, say, z, thus endowing it with a $R^1 \times R^{d-1} \times S^1$ topology. We decompose $\mathbf{k}$ into a component along the periodic dimension and call the remaining components $\mathbf{k}_\perp$:

$$
\mathbf{k} = \left(\mathbf{k}_\perp, \frac{2\pi n}{L}\right) = (\mathbf{k}_\perp, ln)\,, l \equiv 2\pi/L \tag{8}
$$
$$
\omega_1 = \sqrt{k_1^2 + l^2\, {n_1}^2} \tag{9}
$$

The normalization and momentum measure are

$$
\int d\mu(\mathbf{k}) = \int_0^\infty k^{d-2}\, dk \int_{S^{d-2}} d\Omega_{d-2} \sum_{n=-\infty}^{\infty} \tag{10}
$$
$$
N_{k_1} = \frac{1}{\sqrt{2^d\, L\, \pi^{d-1}\, \omega_1}} \tag{11}
$$

With this Phillips and Hu [28] obtained an expression for the energy density fluctuations by integrating the noise kernel (a distribution) over a smearing field, with suitable regularization. From it they obtained for the

Casimir geometry with $R^1 \times R^{d-1} \times S^1$ topology the quantity Δ which is the ratio of the regularized energy density fluctuations to the mean:

$$\Delta' = \frac{\langle\rho^2\rangle - \langle\rho\rangle^2}{\langle\rho\rangle^2}.$$

an alternative definition of the ratio of fluctuations to the mean:

$$\Delta'(\Sigma = R^d \times S^1) = \frac{(d+1)(d+2)}{2}.$$

$$\Delta_{L,\mathrm{Reg}} \equiv \frac{\Delta\rho^2_{L,\mathrm{Reg}}}{\Delta\rho^2_{L,\mathrm{Reg}} + (\rho_{L,\mathrm{Reg}})^2} = \frac{d\,(d+1)}{2+d+d^2} \tag{12}$$

Note the values:

d	1	3	5	∞
$\Delta_{L,\mathrm{Reg}}$	$\frac{1}{2}$	$\frac{6}{7}$	$\frac{15}{16}$	1

3 Noise kernel and heat capacity in Einstein universe

The metric of the Einstein universe with topology $R^1 \times S^3$is

$$ds^2_{Ein} = -dt^2 + a^2 d\Omega^2_{S3}\ , \tag{13}$$

where a is the radius of S^3. The line element $d\Omega^2_{S3}$ of a 3-sphere is given by

$$d\Omega^2_{S3} = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\ , \tag{14}$$

where χ, θ, ϕ are the angular coordinates on S^3 with ranges $0 \le \chi, \theta \le \pi,\ 0 \le \phi \le 2\pi$.

Using the zeta function regularization Phillips and Hu [35] calculated the variance of the energy density of a massless conformally coupled scalar field on the Einstein Universe to be:

$$\Delta\rho^2 = \frac{37}{76800\pi^4 a^8}. \tag{15}$$

Using the result [?] for the energy density,

$$\rho = \frac{1}{480\pi^2 a^4} \tag{16}$$

they find the dimensionless measure of the fluctuations in the energy density (??) is give $\Delta' = \frac{\langle\rho^2\rangle - \langle\rho\rangle^2}{\langle\rho\rangle^2}$

$$\Delta'(\Sigma = S^3) = 111. \tag{17}$$

Thus for the Einstein Universe, the fluctuations in the energy density, like the previous cases, again are of the order unity. Effects due to the fluctuations of the metric will become important *long before* the size of the Universe approaches that of the Planck scale.

Vacuum Fluctuations in Einstein Cylinder S1xR1

$$\langle T_{xx}(\tau, x)\rangle = \frac{\pi}{L^2}\sum_{n_x}|n_x| = -\frac{\pi}{6L^2}$$

with the pressure derived from the Casimir energy

$$P = -\frac{\partial E}{\partial L} = -\frac{\pi}{6L^2}$$

e the compressibility κ.

$$\kappa = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right) = -\frac{1}{L}\left(\frac{\pi}{3L^3}\right)^{-1} = -\frac{3L^2}{\pi}$$

We shall consider a thermal minimally coupled massless scalar field in the Einstein cylinder with periodic boundary condition. To derive the Helmholtz free energy we start with the partition function, with the notation of Phillips and Hu [53],

S1 x S1

$$Z = e^{W} \tag{1}$$

where using the proper-time zeta function method,

$$\begin{aligned} W &= -\frac{1}{2}\text{Tr}\ln\left(\frac{H}{\mu}\right) \\ &= \lim_{s\to 0}\frac{1}{2}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\,\text{Tr}(e^{-tH})\right] \\ &= \lim_{s\to 0}\frac{1}{2}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_n e^{-t\lambda_n}\right] \end{aligned} \tag{2}$$

with the operator

$$H = -\Box = -\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial x^2} \tag{3}$$

where τ is the Euclidean time, the eigenvalues

$$\lambda_{n_0,n} = k_0^2 + k^2 \quad ; \quad k_0 = \frac{2\pi n_0}{\beta} \quad ; \quad k = \frac{2\pi n}{L} \tag{4}$$

and the eigenfunctions

$$\phi_{n_0,n}(x) = \left(\frac{1}{\sqrt{\beta}}e^{ik_0\tau}\right)\left(\frac{1}{\sqrt{L}}e^{ikx}\right). \tag{5}$$

With the partition function Z, we proceed to calculate the free energy F.

$$\begin{aligned} Z = e^{-\beta F} \Rightarrow F &= -\frac{W}{\beta} \\ &= \left(-\frac{1}{2\beta}\right)\lim_{s\to 0}\frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\sum_{n_0,n} e^{-tk_0^2}e^{-tk^2}\right] \end{aligned} \tag{6}$$

With the free energy we are able to derive various thermodynamic quantities including the heat capacities and the compressibilities [79] in the low temperature limit. First, the entropy

$$S = 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots . \tag{15}$$

With the entropy S, we have the internal energy

$$E = F + TS = -\frac{\pi}{6L} + \frac{1}{\beta} + \cdots, \tag{16}$$

which is independent of the scale μ. The dominant term of the internal energy comes from the Casimir effect as given by Equation (12). The subdominant term, which is linear in T, is the zero mode contribution. The corresponding energy density would then be

$$\rho = \frac{E}{L} = -\frac{\pi}{6L^2} + \frac{1}{\beta L} + \cdots. \tag{17}$$

From the free energy, pressure P is just

$$P = -\left(\frac{\partial F}{\partial L}\right)_T = -\frac{\pi}{6L^2} + \cdots. \tag{18}$$

We thus have negative pressure coming from the Casimir effect. The magnitude increases as the size L gets smaller. Negative pressure would shrink the spatial circle and this shrinking force would get larger as the size of circle gets smaller. This process is very similar to that of a gravitational collapse. In fact, we shall see in the subsequent discussions, the Casimir effects of spatial spheres with different dimensions would all induce negative pressure and same kind of collpases should therefore occur.

From the entropy, we obtain the heat capacity at constant volume which is given by the second temperature derivative of the Helmholtz free energy:

$$C_V = -\beta \left(\frac{\partial S}{\partial \beta} \right)_L = 1 + \cdots \tag{19}$$

Hence, other than exponentially small terms, C_V is basically a constant.

Furthermore, the second derivative of the free energy with respect to volume L is related to the isothermal compressibility κ_T,

$$\kappa_T = -\frac{1}{L} \left(\frac{\partial L}{\partial P} \right) = -\frac{3L^2}{\pi} + \cdots . \tag{20}$$

κ_T is negative due to the negative pressure of the Casimir effect. This means that when pressure is increased, or the magnitude of the pressure is decreased, the volume L will increase. This is in contrast to the case of a normal gas where the volume would decrease when the pressure is increased with positive compressibility.

Using the thermodynamic quantities we have obtained, we can also derive the heat capacity at constant pressure C_P. By applying the cyclic and the Maxwell relations, the following relation can be established [80].

$$C_P = C_V + \beta^3 L \kappa_T \left(\frac{\partial P}{\partial \beta}\right)^2 = 1 + \cdots . \tag{27}$$

Since $\partial P/\partial\beta$ is exponentially small, we can see that $C_P \sim C_V$ up to order $e^{-2\pi\beta/L}$. Finally one can also obtain the adiabatic compressibility

$$\kappa_S = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right)_S \tag{28}$$

Again using the Maxwell and the cyclic relations, we have the identity [80]

$$\boxed{\frac{\kappa_S}{\kappa_T} = \frac{C_V}{C_P}} \Rightarrow \kappa_S = \left(\frac{C_V}{C_P}\right)\kappa_T. \tag{29}$$

As we have seen that $C_P \sim C_V$ up to order $e^{-2\pi\beta/L}$. Therefore, we also have $\kappa_S \sim \kappa_T$ other than exponentially small terms.

$$\kappa_S = -\frac{1}{L}\left(\frac{\partial L}{\partial P}\right) = -\frac{3L^2}{\pi} + \cdots . \tag{30}$$

Note that for a closed system, the thermodynamic processes would be adiabatic. Here, κ_S being negative would induce the kind of collapses we mentioned above for a closed spatial geometry. In the subsequent sections, we shall see that this is true for the cases of different spatial geometries in the low temperature expansion.

Now, Fluctuations:

Landau-Lifshitz fluctuations relations do not include vacuum fluctuations which are of more interest to us

With the considerations above, it is also possible to establish the fluctuations of various thermodynamic quantities. Consider a small part of an equilibrium system. Assume that the small part is still large enough for the thermodynamic limit to hold. According to the fluctuation theory of Landau and Lifshitz [80,81], one can then derive the fluctuations of the various thermodynamic quantities in this small part. Using the temperature T and the volume L as independent variables, we have the mean square fluctuation of the temperature

$$\langle (\Delta T)^2 \rangle = \frac{T^2}{C_V} = \frac{1}{\beta^2 C_V}, \tag{31}$$

while the fluctuation of the volume

$$\langle (\Delta L)^2 \rangle = LT|\kappa_T| = \frac{L|\kappa_T|}{\beta}. \tag{32}$$

As the fluctuation should be a positive quantity and in our case κ_T is actually negative, we therefore define the fluctuation to be related to the absolute value of κ_T instead.

Then the fluctuation of the internal energy is given by

$$\begin{aligned} \langle (\Delta E)^2 \rangle &= \frac{C_V}{\beta^2} + \frac{L|\kappa_T|}{\beta} \left[-\beta \frac{\partial P}{\partial \beta} - P \right]^2 \\ &= \frac{\pi}{12\beta L} + \frac{1}{\beta^2} + \cdots \end{aligned} \tag{33}$$

which is proportional to T. Note that as C_V is a constant and $\partial P/\partial \beta$ are exponentially small, the leading contribution comes from $L|\kappa_T|P^2/\beta$.

Again according to the Landau–Lifshitz fluctuation theory, the fluctuation of the pressure is given by

$$\langle (\Delta P)^2 \rangle = \frac{1}{\beta L|\kappa_S|} = \frac{\pi}{3\beta L^3} + \cdots \tag{34}$$

which is also proportional to T.

Moreover, the correlated fluctuation is then

$$\langle (\Delta E)(\Delta P) \rangle = \frac{P}{\beta} = -\frac{\pi}{6\beta L^2} + \cdots \tag{35}$$

We can see that the leading behaviors of these fluctuations are all proportional to T. In other words, as $T \to 0$, all the fluctuations will vanish. It is therefore apparent that the fluctuations derived above are all of thermal nature. No quantum fluctuations are included in Landau and Lifshitz's theory of fluctuations.

3. 4d: Einstein Universe $S^1 \times S^3$

For thermal fields in the Einstein universe, the topology of spacetime is $S^1 \times S^3$. The metric can be written as

$$ds^2 = d\tau^2 + a^2 d\bar{\Omega}_3^2 \tag{55}$$

where $\bar{\Omega}_3$ is the solid angle of a three sphere, and a is the "radius" characterizing the size of the sphere. The operator H in Equation (2) is now given by

$$H = -\frac{\partial^2}{\partial\tau^2} - \frac{1}{a^2}\bar{\Box}, \tag{56}$$

where $\bar{\Box}$ is the Laplacian on S^3. The eigenvalue of $\bar{\Box}$ on S^3 is [82]

$$\bar{\lambda}_n = -n(n+2), \tag{57}$$

with degeneracy

$$\bar{D}_n = (n+1)^2. \tag{58}$$

Hence, the eigenvalue of H is

$$\lambda = k_0^2 + \left(\frac{1}{a^2}\right) n(n+2). \tag{59}$$

The corresponding eigenfunctions are

$$\phi_{n_0,n}(x) = \left(\frac{1}{\sqrt{\beta}} e^{ik_0\tau}\right)\left(\frac{1}{\sqrt{a^3}} Y_3(\Omega)\right), \tag{60}$$

where $Y_3(\Omega)$ is the hyperspherical harmonics on S^3 [83]. Using the eigenvalues, one can establish the free energy, similar to Equation (6), for the Einstein universe as

$$F = -\frac{1}{2\beta} \lim_{s\to 0} \frac{d}{ds}\left[\frac{\mu^s}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \sum_{n_0=-\infty}^{\infty} e^{-t\left(\frac{2\pi n_0}{\beta}\right)^2} \sum_{n=0}^{\infty} (n+1)^2 e^{-\left(\frac{t}{a^2}\right) n(n+2)}\right] \tag{61}$$

$$F = -\frac{0.224909}{a} - \frac{1}{32a}\ln(\mu a^2) + \frac{1}{2\beta}\ln(\mu\beta^2) + \cdots \tag{70}$$

where the ellipsis represents exponentially small terms.

With the above Helmholtz free energy in the low temperature expansion, we derive the various thermodynamic quantities. For the entropy

$$S = 1 - \frac{1}{2}\ln(\mu\beta^2) + \cdots \tag{71}$$

which is the entropy of the zero mode with the same expression as in Equation (15).

For the internal energy

$$E = -\frac{0.224909}{a} - \frac{1}{32a}\ln(\mu a^2) + \frac{1}{\beta} + \cdots, \tag{72}$$

and the energy density, with the volume of the three sphere $V = 4\pi^2 a^3$,

$$\rho = \frac{E}{V} = -\frac{0.00570}{a^4} - \frac{1}{128\pi^2 a^4}\ln(\mu a^2) + \frac{1}{4\pi^2 a^3 \beta} + \cdots. \tag{73}$$

The leading behavior in the low temperature expansion comes from the first two terms. They constitute the Casimir contribution which is μ dependent. Moreover, the pressure

$$P = -\frac{0.00274258}{a^4} - \frac{1}{192\pi^2 a^4}\ln(\mu a^2) + \cdots \tag{74}$$

which is negative due to the Casimir terms and is also dependent on μ.

From the entropy S, we derive the heat capacity at constant volume

$$C_V = 1 + \cdots \tag{75}$$

which is basically a constant with the temperature terms exponentially suppressed. From the pressure P, we obtain the isothermal compressibility

$$\kappa_T = -a^4 \left[0.00330496 + \frac{1}{144\pi^2} \ln(\mu a^2) \right]^{-1} + \cdots \tag{76}$$

which is negative and dependent on μ. Again, the temperature-dependent terms are exponentially suppressed. Since the temperature-dependent terms in P are exponentially small, the thermal expansion coefficient α which depends on $\partial P / \partial T$ is therefore exponentially small too. For the same reason, the heat capacity at constant pressure $C_P \sim C_V$ up to exponentially small terms. The same applies to the two compressibilities $\kappa_S \sim \kappa_T$.

With C_V and κ_T, we can derive the fluctuation of the internal energy

$$\langle (\Delta E)^2 \rangle = \frac{2\pi^2}{\beta a} \left[\frac{(0.00274258 + 0.000527714 \ln(\mu a^2))^2}{0.00330496 + 0.000703619 \ln(\mu a^2)} \right] + \frac{1}{\beta^2} + \cdots \tag{77}$$

which is proportional T. Note that since κ_T is negative and the fluctuation should be a positive quantity, we have taken the absolute value of κ_T in the formula for $\langle (\Delta E)^2 \rangle$. The fluctuation of P is related to the adiabatic compressibility κ_S. Again we take its absolute value in the formula.

$$\langle (\Delta P)^2 \rangle = \frac{1}{\beta a^7} \left[0.000167431 + 0.0000356458 \ln(\mu a^2) \right] + \cdots \tag{78}$$

which is also proportional to T and dependent on the renormalization scale. Moreover, the correlation between the fluctuations of E and P,

$$\langle (\Delta E)(\Delta P) \rangle = -\frac{1}{\beta a^4} \left[0.00274258 + 0.000527714 \ln(\mu a^2) \right] + \cdots \tag{79}$$

- All the thermodynamic quantities and their fluctuations depend on the renormalization scale μ involving divergent quantities. A renormalization procedure is needed to define the various physically measurable quantities.

- Actually, the dependence on μ comes from the fact that we have an odd-dimensional sphere (S3) here. This is not the case with even spatial S2 $_{2}$and S4 dimensional spheres.

Dynamical Spacetimes

- For **a system in expansion or contraction** (like the Universe) thermal equilibrium is not automatic. In fact, **particle creation will constantly disrupt the equilibrium condition** and a general treatment requires **nonequilibriuum quantum field theory.**
- However, there is a special class of **exponential expansion** where particle created has a **thermal spectrum.** For this class of spacetimes which include the **inflationary universe** we can use the **thermal field theory method** established here to investigate the **quantum thermodynamics of dynamical spacetimes** such as calculating their **heat capacity and quantum compressibility**

II. THERMAL PARTICLE CREATION IN COSMOLOGICAL SPACETIMES: EXPONENTIAL SCALING

Consider a spatially flat ($k=0$) Robertson-Walker (RW) universe with the metric

$$ds^2 = dt^2 - a^2 \sum_i (dx^i)^2, \qquad (2.1)$$

where t is cosmic time. A conformally coupled massive (m) scalar field Φ obeys the wave equation (e.g., [30])

$$[\Box + m^2 + R/6]\Phi(t,x) = 0, \qquad (2.2)$$

where $\Box$ is the Laplace-Beltrami operator, and $R = 6[\ddot{a}/a + (\dot{a}/a)^2]$ is the curvature scalar. In a spatially homogeneous space, the space and time parts of the wave function separate, with mode decomposition $\Phi(t,x) = \Sigma_k \phi_k(t) w_k(x)$. For a spatially flat RW universe $w_k(x) = e^{ikx}$, and the conformally related amplitude function $\chi_k(\eta) = a\,\phi_k(t)$ of the k^{th} mode obeys the wave equation in conformal time $\eta = \int dt/a$:

$$\chi_k(\eta)'' + [k^2 + m^2 a^2(\eta)]\chi_k(\eta) = 0. \qquad (2.3)$$

Call $\Phi_k^{\text{in,out}}(t,x)$ the modes with only positive frequency components at $t_- = -\infty$ and $t_+ = +\infty$, respectively. They are related by the Bogolubov coefficients α_k, β_k as follows:

$$\Phi_k^{\text{in}}(t,x) = \alpha_k \Phi_k^{\text{out}}(t,x) + \beta_k \Phi_{-k}^{\text{out}*}(t,x). \qquad (2.4)$$

[For conformal fields it is convenient to use the conformally related wave function $X(\eta,x) = a\Phi(t,x)$. One can define the conformal vacua at $\eta_\pm$ with $\chi^{\text{in, out}}$ in terms of the positive frequency components.] The modulus of their ratio is useful for calculating the probability $P_n(\vec{k})$ of observing n particles in mode $\vec{k}$ at late times [14]:

$$P_n(\vec{k}) = |\beta_k / \alpha_k|^{2n} |\alpha_k|^{-2}. \qquad (2.5)$$

From this one can find the average number of particles $\langle N_{\vec{k}} \rangle$ created in mode $\vec{k}$ (in a comoving volume) at late times to be

$$n_k \equiv \langle N_k \rangle = \sum_{n=0}^{\infty} n P_n(\vec{k}) = |\beta_k|^2. \qquad (2.6)$$

A. Bernard-Duncan model

In a model studied by Bernard and Duncan [16] the scale factor $a(\eta)$ evolves like

$$\text{case 1:} \quad \boxed{a^2(\eta) = A + B \tanh \rho\, \eta,} \tag{2.7}$$

which tends to constant values $a_{\pm}^2 \equiv A \pm B$ at asymptotic times $\eta \rightarrow \pm\infty$. Here ρ measures how fast the scale factor rises, and is the relevant parameter which enters in the temperature of thermal radiance. With this form for the scale function, α_k, β_k have analytic forms in terms of products of gamma functions. One obtains

$$|\beta_k/\alpha_k|^2 = \sinh^2(\pi\omega_-/\rho)/\sinh^2(\pi\omega_+/\rho), \tag{2.8}$$

where

$$\omega_{\pm} = \frac{1}{2}(\omega^{\text{out}} \pm \omega_{\text{in}}), \quad \omega_{\text{in}}^{\text{out}} = \sqrt{k^2 + m^2 a_{\pm}^2}. \tag{2.9}$$

For cosmological models in which $a(+\infty) \gg a(-\infty)$, the argument of sinh is very large [i.e., $(\pi/\rho)\omega_{\pm} \gg 1$]. To a good approximation this has the form

$$|\beta_k/\alpha_k|^2 = \exp(-2\pi\omega_{in}/\rho). \quad (2.10)$$

For high momentum modes, one can recognize the Planckian distribution with temperature given by

$$k_B T_\eta = \rho/(2\pi a_+) \quad (2.11)$$

as detected by an observer (here in the conformal vacuum) at late times.

Thermal Particle Creation from Exponentially expanding universe

- Consider a massive conformal scalar field in a closed 4D FLRW universe

$$ds^2 = -dt^2 + a^2(t)\, d\Omega_{S^3}^2 = a^2(\eta)\,(-d\eta^2 + d\Omega_{S^3}^2)$$

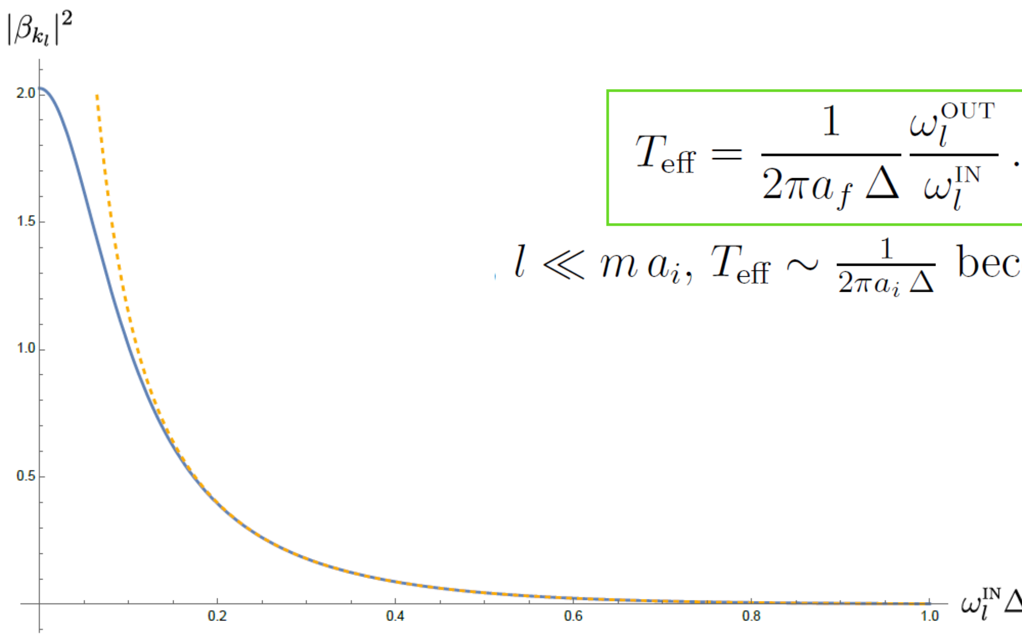
with the radius a evolving in conformal time η

$$a^2(\eta) = \frac{a_f^2 + a_i^2}{2} + \frac{a_f^2 - a_i^2}{2} \tanh\frac{\eta}{\Delta}$$

In a statically-bounded setup. Note in the exponential rise regime the transit duration Δ is the inverse of the rise rate ρ.

Vacuum fluctuations in the in-vacuum is parametrically amplified by the expansion of the universe, and with an initial period of exponential expansion – inflationary universe– particles produced have a thermal spectrum.

Thermal particle production



$$T_{\text{eff}} = \frac{1}{2\pi a_f\,\Delta}\,\frac{\omega_l^{\mathrm{OUT}}}{\omega_l^{\mathrm{IN}}}\,.$$

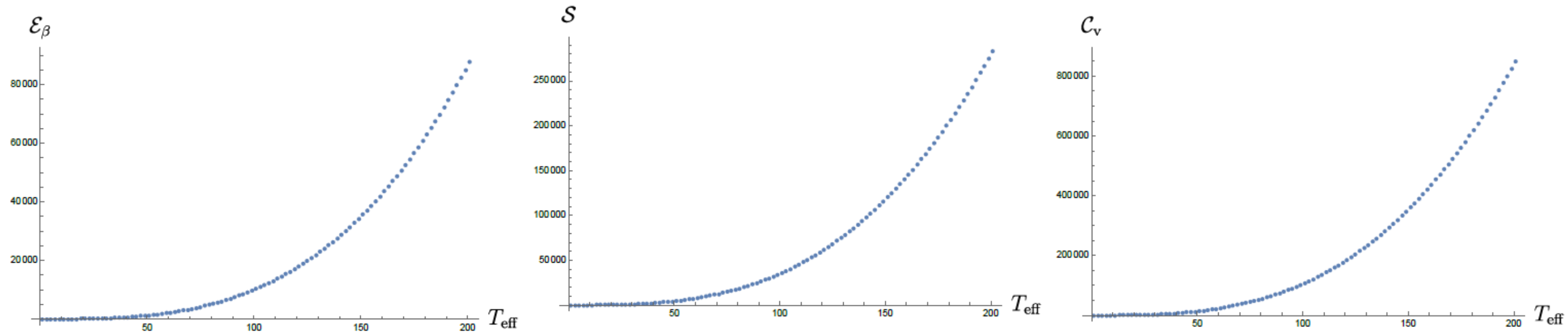
$l \ll m\,a_i$, $T_{\text{eff}} \sim \frac{1}{2\pi a_i\,\Delta}$ becomes

FIG. 1. dependence of $|\beta_{k_l}|^2$ on Δ is shown in the blue curve. In comparison, the original dashed curve is the corresponding result according to the Bose-Einstein distribution. They deviate for very small Δ, and it is the consequence of Eq. (28). Here we choose $\omega_l^{\mathrm{OUT}} = 10$ and $\omega_l^{\mathrm{IN}} = 1$.

$$E = \sum_{l=0}^{\infty} d_l^{(4)} \left(|\beta_{k_l}|^2 + \frac{1}{2} \right) \varpi_l$$

$d_l^{(n+1)}$ in $R \times S^n$ is degeneracy

$$d_l^{(n+1)} = (2l + n - 1) \frac{(l + n - 2)!}{l!(n-1)!} = (l+1)^2 ,$$



(a) energy correction $\mathcal{E}_\beta$ (b) von Neumann entropy $\mathcal{S}$ (c) heat capacity $\mathcal{C}_{\rm V}$

FIG. 4. Here we show $\mathcal{E}_\beta$, $\mathcal{S}$, and $\mathcal{C}_{\rm V}$ of the whole field after we add up the contribution of each mode. Though they look similar, in fact $\mathcal{E}_\beta$ like $T_{\rm eff}^4$ while $\mathcal{S}$, and $\mathcal{C}_{\rm V}$ grow like $T_{\rm eff}^3$ for sufficiently large $T_{\rm eff}$.

physical frequency is $\varpi_l = \frac{\omega_l}{a} = \sqrt{\frac{(l+1)^2}{a^2} + m^2}.$ d_l is degeneracy in level l

$$\mathcal{Z} = \prod_{l=0}^{\infty} \left[\sum_{N_l=0}^{\infty} e^{-(N_l+1/2)\beta_{\text{eff}}\varpi_l} \right]^{d_l^{(4)}} = \prod_{l=0}^{\infty} \left[\frac{e^{\frac{\beta_{\text{eff}}\varpi_l}{2}}}{e^{\beta_{\text{eff}}\varpi_l} - 1} \right]^{d_l^{(4)}} .$$

$$\mathcal{F} = -\frac{1}{\beta_{\text{eff}}} \ln \mathcal{Z} = \sum_{l=0}^{\infty} d_l^{(4)} \left[\frac{\varpi_l}{2} + \frac{1}{\beta_{\text{eff}}} \ln\left(1 - e^{-\beta_{\text{eff}}\varpi_l}\right) \right] .$$

$$\mathcal{E} = -\frac{\partial}{\partial \beta_{\text{eff}}} \ln \mathcal{Z} = \sum_{l=0}^{\infty} d_l^{(4)} \frac{\varpi_l}{2} \coth \frac{\beta_{\text{eff}}\varpi_l}{2} ,$$

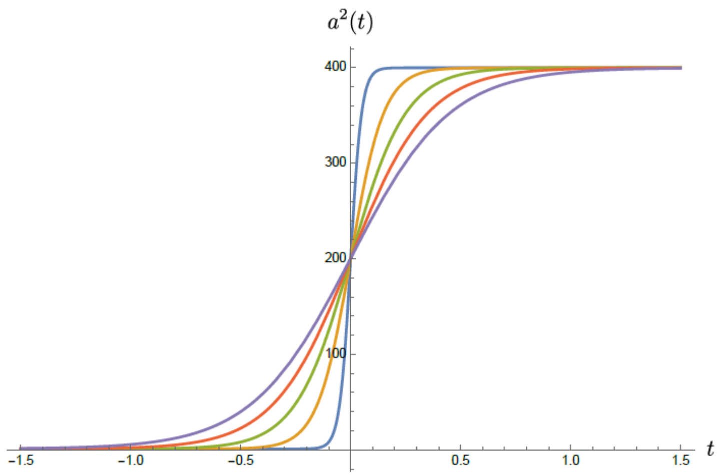
$$\mathcal{S} = \beta_{\text{eff}}^2 \left(\frac{\partial \mathcal{F}}{\partial \beta_{\text{eff}}} \right)_V = \sum_{l=0}^{\infty} d_l^{(4)} \left[\frac{\beta_{\text{eff}}\,\varpi_l}{e^{\beta_{\text{eff}}\varpi_l} - 1} - \ln\left(1 - e^{-\beta_{\text{eff}}\varpi_l}\right) \right]$$

$$\mathcal{C}_{\text{V}} = -\beta_{\text{eff}}^2 \left(\frac{\partial \mathcal{E}}{\partial \beta_{\text{eff}}} \right)_V = -\sum_{l=0}^{\infty} d_l^{(4)} \frac{\beta_{\text{eff}}^2 \varpi_l^2}{4} \operatorname{csch}^2 \frac{\beta_{\text{eff}}\varpi_l}{2} .$$

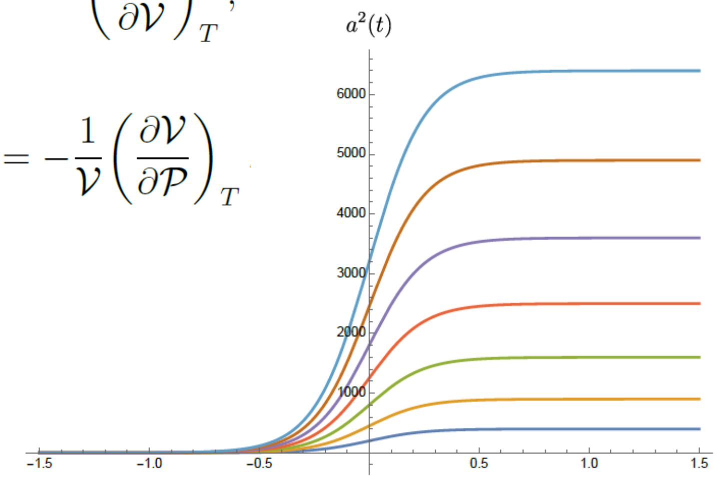
Varying T (for C_V) and V (for P_T) 1-parameter family of solutions

$$\mathcal{P} = -\left(\frac{\partial \mathcal{F}}{\partial \mathcal{V}}\right)_T ,$$

$$\kappa_{\mathrm{T}} = -\frac{1}{\mathcal{V}}\left(\frac{\partial \mathcal{V}}{\partial \mathcal{P}}\right)_T$$



(a) fixed a_f



(b) fixed Δ

FIG. 5. Here we show the behavior of $a^2(t)$ for the different scenarios we consider. In (a), we fixed the final scale factor a_f but varies Δ, so that we can compare the results for different effective temperatures. In (b) we fix Δ, but allow a_f to take different values. Thus we keep the temperature constant but change the volume.

General Dynamical Spacetimes

- Particle creation not necessarily thermal,
  - cannot assume thermal equilibrium and treat it like thermal field in static space.

* Particle creation in real time evolution:

Vacuum: Numerical computation of particle number, energy and pressure density, from which, calculate vacuum compressibility Yucun Xie, J. T. Hsiang and B. L. Hu, **Dynamical Vacuum Compressibility of Space** Phys. Rev. D109, 065027 (2024)

Better, In-In calculation using the newly discovered nonequilibrium free energy density functional

Nonequilibrium Quantum Free Energy and Effective Temperature, Generating Functional and Influence Action J. T. Hsiang and B. L. Hu, Phys. Rev. D103, 065001 (2021) 10.1103/PhysRevD.103.065001

3 kinds of vacuum processes: Casimir Effect, Trace Anomaly & Particle Creation

TABLE I. The red markings (left column) are used to indicate quantum effects due to a massless conformal field, while blue markings (right column) are for a massive conformal field. A check mark indicates existence and a cross mark indicates absence.

	$\mathbf{R} \times S^1$	$\mathbf{R} \times S^2$	$\mathbf{R} \times S^3$	$\mathbf{R} \times T^3$
Casimir energy	✓ ✓	× ✓	✓ ✓	✓ ✓
Trace anomaly	✓ ✓	× ×	✓ ✓	✓ ✓
Particle production	× ✓	× ✓	× ✓	✓ ✓

Conformal Stress Tensor

The classical energy-momentum tensor of this scalar field is given by

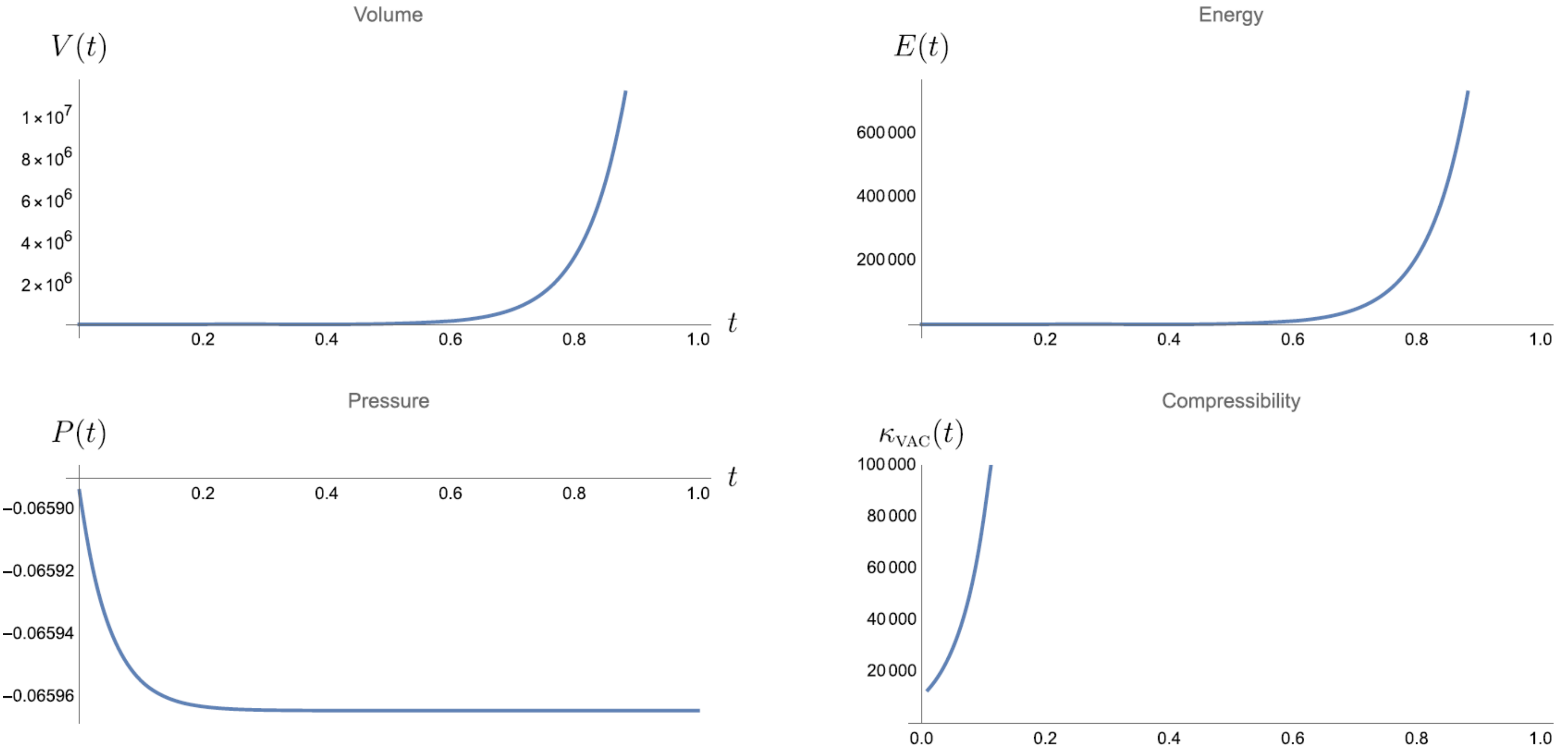
$$T_{\mu\nu} = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} g_{\mu\nu} g^{\lambda\sigma} (\partial_\lambda \phi)(\partial_\sigma \phi) - \frac{1}{6} [\nabla_\mu \partial_\nu (\phi^2) - g_{\mu\nu} g^{\lambda\sigma} \nabla_\lambda \partial_\sigma (\phi^2) + \phi^2 G_{\mu\nu}], \qquad (64)$$

where $G_{\mu\nu}$ is the Einstein tensor and ∇_μ is the covariant derivative with respect to the metric tensor $g_{\mu\nu}$. The

S^3: massless conformal scalar field: Casimir Effect and Trace Anomaly

a (t) ~ e^{Ht}

$$= \frac{1}{480\pi^2 a^2}\left\{\frac{1}{a^2} + \frac{\dot{a}^2\ddot{a}}{a} - \frac{\dot{a}^4}{a^2} - \frac{\ddot{a}^2}{2} + \dot{a}\dddot{a}\right\}.$$



T^3: Massive conformal field: Casimir Effect and Particle Creation

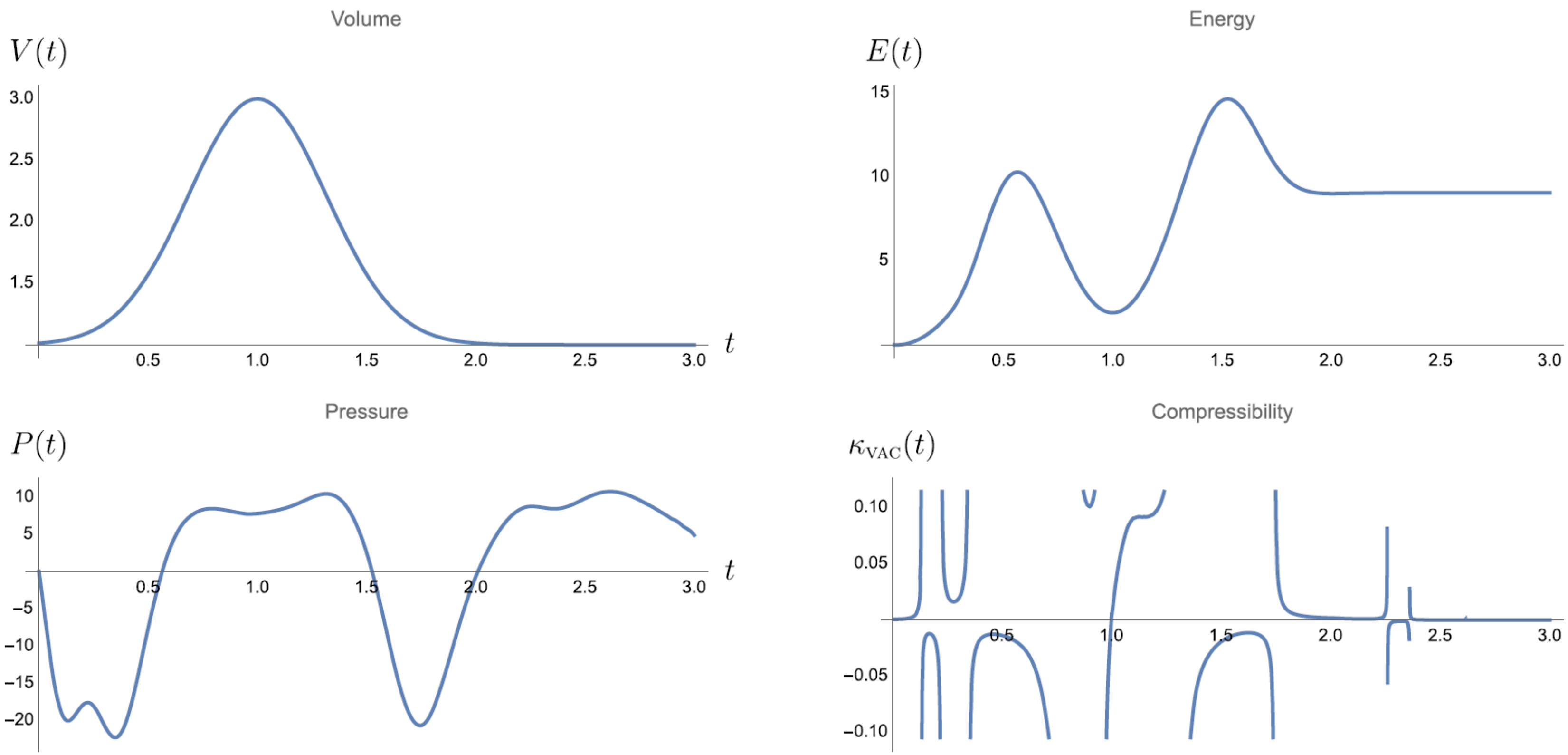


FIG. 14. The volume, pressure, energy, and vacuum compressibility of a massless conformal field in a rectangular T^3 box when one side of the box is allowed to expand and contract in cosmic time. The scalar factor follows a Gaussian evolution Eq. (20), with $a_0 = 1$, $\Delta = 2$, $t_0 = 1$ and $1/\tau^2 = 5$.

A bit crazy, a lot to chew on!

Next Stage of Work:

- Calculate the thermodynamic quantities from the stress energy tensor and its fluctuations, by way of the noise kernels.
- This will impart quantum thermodynamic meanings to the quantum field theoretical quantities of curved spacetime QFT.
- As an example we have calculated the expression for the fluctuations of the Txx component of the stress energy tensor for the Einstein universe.
- Connect that expression to the adiabatic compressibility.
- We can then better appreciate the quantum thermodynamic properties of the noise kernel, such as the quantum capacity and the vacuum compressibility of spacetime.
- Vacuum fluctuation phenomena are best described by stochastic gravity, with noise kernel as its centerpiece. Connecting with quantum thermodynamics will impart useful physical meaning to this theory.

Thank You!

Phillips and Hu, 1997

$$\Delta T^2_{\mu\nu\alpha\beta}(x,x') = \lim_{s,t\to 0} \frac{1}{2}\frac{d}{ds}\Bigg\{\frac{\mu^s}{\Gamma(s)}\int_0^\infty du \int_0^\infty dv\,(u+v)^s(uv)^t$$
$$\sum_{nn'} e^{-u\lambda_n - v\lambda_{n'}} T_{\mu\nu}[\phi_n(x),\phi^*_{n'}(x)]T_{\alpha\beta}[\phi_{n'}(x'),\phi^*_n(x')]\Bigg\} \quad (21)$$

We use the stress tensor expression in Eq. (7) again here and take the $(\tau', x') \to (\tau, x)$ limit. Then,

$$\Delta T^2_{xxxx}(\tau,x;\tau,x) = \lim_{s,t\to 0}\left(\frac{1}{2\beta^2 L^2}\right)\frac{d}{ds}\Bigg\{\frac{\mu^s}{\Gamma(s)}\int_0^\infty du \int_0^\infty dv\,(u+v)^s(uv)^t$$

Pressure fluctuations

$$\sum_{nn'} e^{-u\lambda_n - v\lambda_{n'}}(k_0'k_0 - kk' - k_0^2 - k^2)(k_0k_0' - k'k - {k_0'}^2 - {k'}^2)\Bigg\} \quad (22)$$

$$\Delta T^2_{xxxx}(\tau,x;\tau,x) = \sum_{n_x,n_x'}\frac{\pi^2|n_x||n_x'|}{L^4}$$
$$= \frac{4\pi^2}{L^4}(\zeta(-1))^2$$
$$= \frac{\pi^2}{36L^4}$$

[check: same as energy density fluctuations for massless fields]

Remaining task: connect with adiabatic compressibility

Summary: Casimir Energy and Pressure Fluctuations, Noise Kernel and Quantum Compressibility of Spacetime

This current research program by Cho, Hsiang and Hu attempts to link up the stress energy fluctuations of quantum fields in spacetimes (with nontrivial topology or curvature) with their thermodynamic properties. The former is represented by the noise kernel, the stress energy tensor correlator of quantum matter fields, while the latter by the heat capacity and the (adiabatic and isothermal) compressibility. We have only touched on the latter part so far. The next stage is to bring in the noise kernel.

Noise kernel is the centerpiece of *stochastic gravity*, a theory for the dynamics of curved spacetimes based on the Einstein-Langevin equation [arXiv:0802.0658] which incorporates fully and self-consistently the backreaction effects from the mean values and the fluctuations of quantum field stress tensors.

Examining the noise kernel from a thermodynamic perspective can add a new dimension to our understanding of its physical properties. E.g., heat capacity at constant volume gives a measure of the fluctuations of the energy density to the mean which provides a criterion for the validity of the canonical distribution.

The intriguing fact coming from the past 3 decades of work is that the fluctuations of energy density to the mean is close to unity for quantum fields in many different spacetimes. From a thermodynamic perspective we conjecture that this feature, even for quantum fields in ordinary Minkowski spacetime, is an indication that the balance between spacetime and quantum matter fields has some built-in thermodynamic instability, that their co-existence meets with a saturation criterion in the "capacity of spacetime" to "hold" the quantum field, a theme we view as worthy of deeper thoughts.

Abstract:

An important yet perplexing result from work in the 1990s and 2000s is the near-unity value of the ratio of fluctuations in the vacuum energy density of quantum fields to the mean in a collection of generic spacetimes.

This was carried out by way of calculating the noise kernels which are the correlators of the stress-energy tensor of quantum fields.

In this paper, we revisit this issue via a quantum thermodynamics approach, by calculating two quintessential thermodynamic quantities: the heat capacity and the quantum compressibility of some model geometries filled with a quantum field at high and low temperatures.

This is because heat capacity at constant volume gives a measure of the fluctuations of the energy density to the mean. When this ratio approaches or exceeds unity, the validity of the canonical distribution is called into question. Likewise, a system's compressibility at constant pressure is a criterion for the validity of grand canonical ensemble.

We derive the free energy density and, from it, obtain the expressions for these two thermodynamic quantities for thermal and quantum fields in 2d Casimir space, 2d Einstein cylinder and 4d (S1 × S3) Einstein universe, as well as S1 × S2 and S1 × S4.

With this array of spacetimes we can investigate the thermodynamic stability of quantum matter fields in them and make some qualitative observations on the compatibility condition for the co-existence between quantum fields and spacetimes, a fundamental issue in the quantum and gravitation conundrum.

B. L. Hu (胡悲樂) **Fluctuations of Quantum Fields and Thermodynamics of Spacetime***

Maryland Center for Fundamental Physics and Joint Quantum Institute, University of Maryland, College Park, USA

My research in the past 40 years can summarily be represented in a 2D phase space with Quantum-Classical on the horizontal axis, and micro-Macro on the vertical. You can tell from the title of my talk that the intention is to **connect the micro features of quantum matter to the macro-structures of spacetime:** matter via quantum field theory, spacetime via general relativity, and micro/macro interface via nonequilibrium statistical mechanics [1]. This stretch may sound wild on surface, but not so surprising if you knew that in my view, **general relativity is in the nature of a hydrodynamic theory** [arXiv:gr-qc/9607070, arXiv:gr-qc/9511077], **valid only in the long wavelength, low energy domain**. GR is a beautiful theory, yet only an effective theory, emergent from **quantum gravity** – theories describing the microscopic structures of spacetime at the Planck length (10^{-35} m), not unlike thermo- and hydro- to molecular-dynamics. To me, hydrodynamics and thermodynamics not only serve as a set of useful tools, but provide the **correct perspective to ask meaningful questions** about the nature and behavior of spacetime as we know it. My recent work with H. T. Cho and J. T. Hsiang attempts to **link up the stress energy fluctuations of quantum fields in spacetimes** (with nontrivial topology or curvature) **with their thermodynamic properties**. The former is represented by the **noise kernel**, the stress energy tensor correlator of quantum matter fields, while the latter by the heat capacity and the (adiabatic and isothermal) compressibility. Noise kernel is the centerpiece of **stochastic gravity** [2], a theory for the dynamics of curved spacetimes based on the Einstein-Langevin equation [arXiv:0802.0658] which incorporates fully and self-consistently the **backreaction effects** from the mean values and the fluctuations of quantum field stress tensors. **Examining the noise kernel from a thermodynamic perspective** can add a new dimension to our understanding of its physical properties. E.g., heat capacity gives a measure of the fluctuations of the energy density to the mean, acting as a criterion for the validity of the canonical distribution. An intriguing fact coming from the past 3 decades of work [3] is that **the fluctuations of energy density to the mean is close to unity for quantum fields in many different spacetimes**. From a thermodynamic perspective we conjecture that this feature, even for quantum fields in ordinary Minkowski spacetime, is an indication that the balance between spacetime and quantum matter fields has some built-in thermodynamic instability, that their co-existence meets with a **saturation criterion in the "capacity of spacetime" to "hold" the quantum field,** a theme we view as worthy of deeper thoughts.

* Based on 1) H. T. Cho, J. T. Hsiang and B. L. Hu, *Quantum Capacity and Vacuum Compressibility of Spacetimes: Thermal Fields*, *Universe 8, 291 (2022).* 2) Yucun Xie, J. T. Hsiang and B. L. Hu, *Dynamical Vacuum Compressibility of Space*, Phys. Rev. D109, 065027 (2024)

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[2] B. L. Hu and E. Verdaguer, *Semiclassical and Stochastic Gravity- Quantum Field Effects on Curved Spacetimes* (Cambridge University Press, Cambridge, 2020). *Liv. Rev. Rel.* **11** (2008) 3

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1. What is Quantum Thermodynamics
2. Q. Thermodynamics of Spacetime: Quantum Fields in Curved Spacetime
3. Energy Density Fluctuations large compared to the mean
4. Stochastic Gravity: Fluctuations
5. Heat Capacity and Quantum Compressibility of Spacetime
6. Static Spaces: Einstein cylinder, S^2, S^3
7. Dynamical Spacetime with thermal particle creation: An easy yet relevant (inflation) case

Quantum Thermodynamics

- Thermodynamics of Quantum Systems
- A new field with many fundamental issues and a wide range of applications.

- Laws of classical thermodynamics apply to large systems and at high temperatures.
- Quantum Devices are getting smaller and better operating at low temperatures.

Why is QTD interesting?

A. Quantum devices: getting smaller, at lower T
B. Rich in many fundamental theoretical issues

Let's look more carefully at the relation between:

- **Thermodynamics (TD) –Stat. Mech (SM)**,
- **Q**uantum –**C**lassical,
- micro (μ) Macro (M)

SM-TD,   Q-C

- **Stat Mechanics is not reliant on Quantum**
- *At the mundane level*: counting of "micro"-states.
  Ping Pong balls in two partitions: Pure statistics.
- When h is treated as the smallest unit of phase space → micro / quantum

- *More sophisticated*: Stat Mech = Eucldiean QFT formally.
  restricted to equilibrium finite T conditions.
- Many quantum geometry calculations involve summing over distinct classical configurations (e.g., geometries, simplices). E.g., Regge Calculus: Kinematics, not dynamics

- **TD=/= classical**. E.g., Quantum fluid hydrodynamics

Classical Equilibrium Thermodynamics can be derived from classical **Statistical Mechanics**

Classical **Equilibrium TD** is the limit of two theories:

- macroscopic limit of **Stat Mechanics** μ–> M
- equilibrium limit of **NEq** Stat. Mech **(NESM)**

- **Validity** of classical **and** quantum **TD** depends on the existence of an equilibrium state.

which needs to be shown via **Neq dynamics**.

- **Equilibration** is a necessary but not sufficient condition for **thermalization.** In short,

> **Quantum TD** cannot be obtained from **QSM** **>>**

.

QTD is **not** a
limiting theory of **QSM**:

Why?

Q in QSM refers only to spin

- **Quantum** -- (A standard course in) **Q stat mech** has not touched the heart of quantum.

  Only treats spin-statistics effects.

QSM Starts with 2 Fundamental Axioms:

1) **Random Phase Approximation**: Q Phase info ignored ab initio. Start from **Probabilities.**
2) **Equal a priori probability** for all accessible states: **Equilibrium** condition

Most important quantum features are lost:

A. Q. **Coherence**: Picked up in Q Optics mid ‘60s

B. Q. **Entanglement**: Quantum Information mid ’90s

This brings out the very important and fundamental issue of

Quantum – Classical Correspondence/ Transition

E.g., **Decoherence –** since the 90s, better understood by now. Many books written.

Definition : QTD = thermodynamics of quantum many-body systems

- **Quantum**: quantum coherence, correlations and entanglements in the system are important, especially at very low T

- **TD** - Is there a **thermodynamic limit** for interacting Q many body systems? What if the system is small to begin with?
- What is large? What is small?

How does one draw the line between μicro-meso-Macro ?

QTD: **T**hermo**D**ynamics

Micro (m) → Macro (M) Aspect

- **TD limit**: large N, V, but finite n=N/V
- How `**large**' is an object, a state space, dof?
- How **big** a composite configuration
  can **TD descriptions** begin to be useful?

- **May not exist for small quantum systems.**
- TD of **small systems** seems a no-go
  but **Q** is for small systems. So,
- **QTD** of **small Q systems** is a new challenge

What are the new challenges?

1. **Equilibrium Thermodynamics in the Q regime**

- Do the thermodynamics quantities, relations and laws *established on the foundation of classical physics still hold for Q systems*? E.g., with
  - strong coupling between system-bath
  - non-Markovian regime: low T, non-Ohmic bath.

2. **Non-Equilibrium dynamics of Open Q System** is a better approach to QTD: you can see the dynamics